

POSET PINBALL, THE DIMENSION PAIR ALGORITHM, AND TYPE A REGULAR NILPOTENT HESSENBERG VARIETIES

DARIUS BAYEGAN AND MEGUMI HARADA

ABSTRACT. In this manuscript we develop the theory of **poset pinball**, a combinatorial game recently introduced by Harada and Tymoczko for the study of the equivariant cohomology rings of GKM-compatible subspaces of GKM spaces. Harada and Tymoczko also prove that in certain circumstances, a **successful outcome of Betti poset pinball** yields a module basis for the equivariant cohomology ring of the GKM-compatible subspace. Our main contributions are twofold. First we construct an algorithm (which we call the **dimension pair algorithm**) which yields the result of a successful outcome of Betti poset pinball for any type A regular nilpotent Hessenberg and any type A nilpotent Springer variety, considered as GKM-compatible subspaces of the flag variety $\mathcal{F}lags(\mathbb{C}^n)$. The definition of the algorithm is motivated by a correspondence between Hessenberg affine cells and certain Schubert polynomials which we learned from Erik Insko. Second, in the special case of the type A regular nilpotent Hessenberg varieties specified by the Hessenberg function $h(1) = h(2) = 3$ and $h(i) = i + 1$ for $3 \leq i \leq n - 1$ and $h(n) = n$, we prove that the pinball result coming from the dimension pair algorithm is **poset-upper-triangular**; by results of Harada and Tymoczko this implies the corresponding equivariant cohomology classes form a $H_{S^1}^*(pt)$ -module basis for the S^1 -equivariant cohomology ring of the Hessenberg variety.

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1. INTRODUCTION

The purpose of this manuscript is to further develop the theory of **poset pinball**, a combinatorial game introduced in [10] for the purpose of computing in equivariant cohomology rings,¹ in certain cases of **type A nilpotent Hessenberg varieties**. One of the main uses of poset pinball in [10] is to construct module bases for the equivariant cohomology rings of **GKM-compatible subspaces** of GKM spaces [10, Definition 4.5]. In the context of this manuscript, the ambient GKM space is the flag variety $\mathcal{F}lags(\mathbb{C}^n)$ equipped with the action of the diagonal subgroup T of $U(n, \mathbb{C})$, and the GKM-compatible subspaces are the nilpotent Hessenberg varieties. It is well-recorded in the literature (e.g. [18] and references therein) that GKM spaces often have geometrically and/or combinatorially natural module bases for their equivariant cohomology rings; the basis of equivariant Schubert classes $\{\sigma_w\}_{w \in S_n}$ for $H_T^*(\mathcal{F}lags(\mathbb{C}^n))$ is a famous example. The results of this manuscript represent first steps towards the larger goal of using poset pinball to construct a similarly computationally effective and convenient module bases for a GKM-compatible subspace by exploiting the structure of the ambient GKM space.

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¹All cohomology rings in this note are with \mathbb{C} coefficients.

We briefly recall the setting of our results. Let $N : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a nilpotent operator. Let $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a function satisfying $h(i) \geq i$ for all $1 \leq i \leq n$ and $h(i+1) \geq h(i)$ for all $1 \leq i < n$. The associated Hessenberg variety $\text{Hess}(N, h)$ is then defined as the following subvariety of $\mathcal{F}lags(\mathbb{C}^n)$:

$$\text{Hess}(N, h) := \{V_\bullet = (0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n) \mid NV_i \subseteq V_{h(i)} \text{ for all } i = 1, \dots, n\}.$$

Since we deal exclusively with type A in this paper, henceforth we omit this phrase from our terminology. Two special cases of Hessenberg varieties are of particular interest in this manuscript: when N is the principal nilpotent operator (in this case $\text{Hess}(N, h)$ is called a **regular nilpotent Hessenberg variety**) and when h is the identity function $h(i) = i$ for all $1 \leq i \leq n$ (in this case $\text{Hess}(N, h)$ is called a **nilpotent Springer variety** and is sometimes denoted \mathcal{S}_N). Hessenberg varieties arise in many areas of mathematics, including geometric representation theory [9, 15, 16], numerical analysis [6], mathematical physics [12, 14], combinatorics [8], and algebraic geometry [4, 5], so it is of interest to explicitly analyze their topology, e.g. the structure of their (equivariant) cohomology rings. We do so through poset pinball and Schubert calculus techniques, as initiated and developed in [1, 10, 11] and briefly recalled below.

The following relationship between two group actions on the nilpotent Hessenberg variety and the flag variety respectively allows us to use the theory of GKM-compatible subspaces and poset pinball. There is a natural S^1 subgroup of the unitary diagonal matrices T which acts on $\text{Hess}(N, h)$ (defined precisely in Section 2). The group T , the maximal torus of $U(n, \mathbb{C})$, acts on $\mathcal{F}lags(\mathbb{C}^n)$ in the standard fashion. It turns out that the S^1 -fixed points $\text{Hess}(N, h)^{S^1}$ are a subset of the T -fixed points $\mathcal{F}lags(\mathbb{C}^n)^T \cong S_n$. Moreover, the inclusion of $\text{Hess}(N, h)$ into $\mathcal{F}lags(\mathbb{C}^n)$ and the inclusion of groups S^1 into T then induces a natural ring homomorphism

$$(1.1) \quad H_T^*(\mathcal{F}lags(\mathbb{C}^n)) \rightarrow H_{S^1}^*(\text{Hess}(N, h)).$$

As mentioned above, it is well-known in Schubert calculus that the equivariant Schubert classes $\{\sigma_w\}_{w \in S_n}$ are a computationally convenient $H_T^*(\text{pt})$ -module basis for $H_T^*(\mathcal{F}lags(\mathbb{C}^n))$. We refer to the images in $H_{S^1}^*(\text{Hess}(N, h))$ of the equivariant Schubert classes $\{\sigma_w\}_{w \in S_n}$ via the projection (1.1) as **Hessenberg Schubert classes**. Given this setup and following [10], the game of poset pinball uses the data of the fixed points $\mathcal{F}lags(\mathbb{C}^n)^T \cong S_n$ (considered as a partially ordered set with respect to Bruhat order) and the subset

$$\text{Hess}(N, h)^{S^1} \subseteq \mathcal{F}lags(\mathbb{C}^n)^T \cong S_n$$

to determine a set of **rolldowns** in S_n . It is shown in [10] that, under certain circumstances (one of which is discussed in more detail below), such a set of rolldowns in turn specifies a subset of the Hessenberg Schubert classes which form a $H_{S^1}^*(\text{pt})$ -**module basis** of $H_{S^1}^*(\text{Hess}(N, h))$. Thus poset pinball is an important tool for building computationally effective module bases for the equivariant cohomology of Hessenberg varieties. Indeed, the results of [11] accomplish precisely this goal – *i.e.* of constructing a module basis via poset pinball techniques – in the special case of the Peterson variety, which is the regular nilpotent Hessenberg variety with Hessenberg function h defined by $h(i) = i + 1$ for $1 \leq i \leq n - 1$ and $h(n) = n$. Exploiting this explicit module basis, in [11, Theorem 6.12] the second author and Tymoczko give a manifestly positive **Monk formula** for the product of a degree-2 Peterson Schubert class with an arbitrary Peterson Schubert class, expressed as a $H_{S^1}^*(\text{pt})$ -linear combination of Peterson Schubert classes. This is an example of equivariant Schubert calculus in the realm of Hessenberg varieties, and it is an open problem to generalize the results of [11] to a wider class of Hessenberg varieties.

We now describe our main results. First, we explain in detail an algorithm which we dub the **dimension pair algorithm** and which associates to each S^1 -fixed point $w \in \text{Hess}(N, h)^{S^1}$ a permutation in S_n , which we call the **rolldown of w** following terminology in [10] and denoted $\text{roll}(w) \in S_n$. In the special cases of regular nilpotent Hessenberg varieties and nilpotent Springer varieties, we show that the set $\{\text{roll}(w)\}_{w \in \text{Hess}(N, h)^{S^1}}$ can be interpreted as the result of a **successful game of Betti pinball** (in the sense of [10]). The main motivation for our construction is that a successful outcome of Betti pinball can, under some circumstances, produce a module basis for the associated equivariant cohomology ring (cf. [10, Section 4.3]). In this sense, our algorithm represents a significant step towards the construction of module bases for the equivariant cohomology rings of general nilpotent Hessenberg varieties, thus extending the theory developed in [10, 11]. Although we formulate our algorithm in terms of dimension pairs and permissible fillings following terminology of Mbirika [13], the essential idea comes from a correspondence between Hessenberg affine cells and certain Schubert polynomials which we learned from Erik Insko.

Second, for a specific case of a regular nilpotent Hessenberg variety which we call a **334-type** Hessenberg variety, we prove that the set of rolldowns $\{\text{roll}(w)\}_{w \in \text{Hess}(N, h)^{S^1}}$ obtained from the dimension pair algorithm is in fact **poset-upper-triangular** in the sense of [10]. As shown in [10], this is one of the possible circumstances under which we can conclude that the corresponding set of Hessenberg Schubert classes forms a module basis for the S^1 -equivariant cohomology ring of the variety. Thus our result gives rise to a new family of examples of Hessenberg varieties (and GKM-compatible subspaces) for which poset pinball successfully produces explicit module bases. We mention that the dimension pair algorithm also produces module bases in a special case of Springer varieties [7]. Although we do not know whether the dimension pair algorithm always succeeds in producing module bases for the S^1 -equivariant cohomology rings for a general nilpotent Hessenberg variety, the evidence thus far is suggestive. We leave further investigation to future work.

We give a brief summary of the contents of this manuscript. In Section 2 we recall some definitions and constructions necessary for later statements. In Section 3.1 we describe the dimension pair algorithm and prove that the result of the algorithm satisfies the conditions to be the outcome of a successful game of Betti poset pinball in the special cases of regular nilpotent Hessenberg varieties and nilpotent Springer varieties. We briefly review in Section 3.2 the theory developed in [10] which show that, if the rolldown set obtained from a successful game of Betti poset pinball also satisfies poset-upper-triangularity conditions, then it yields a module basis in equivariant cohomology. In Sections 4 and 5 we prove that the dimension pair algorithm produces a poset-upper-triangular module basis in a special class of regular nilpotent Hessenberg varieties which we call 334-type Hessenberg varieties. We close with some open questions in Section 6.

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2. BACKGROUND

We begin with necessary definitions and terminology for what follows. In Section 2.1 we recall the geometric objects and the group actions under consideration. In Section 2.2 we recall some combinatorial definitions associated to Young diagrams. We recall a bijection between Hessenberg fixed points and certain fillings of Young diagrams in Section 2.3. The discussion closely follows previous work (e.g. [10, 11] and also [17]) so we keep exposition brief.

2.1. Hessenberg varieties, highest forms, and fixed points. By the **flag variety** we mean the homogeneous space $GL(n, \mathbb{C})/B$ which is also identified with

$$\mathcal{Flags}(\mathbb{C}^n) := \{V_\bullet = (\{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i\}.$$

A **Hessenberg function** is a function $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ satisfying $h(i) \geq i$ for all $1 \leq i \leq n$ and $h(i+1) \geq h(i)$ for all $1 \leq i < n$. We frequently denote a Hessenberg function by listing its values in sequence, $h = (h(1), h(2), \dots, h(n) = n)$. Let $N : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear operator. The **Hessenberg variety** $\text{Hess}(N, h)$ is defined as the following subvariety of $\mathcal{Flags}(\mathbb{C}^n)$:

$$(2.1) \quad \text{Hess}(N, h) := \{V_\bullet \in \mathcal{Flags}(\mathbb{C}^n) \mid NV_i \subseteq V_{h(i)} \text{ for all } i = 1, \dots, n\} \subseteq \mathcal{Flags}(\mathbb{C}^n).$$

If N is nilpotent, we say $\text{Hess}(N, h)$ is a **nilpotent Hessenberg variety**, and if N is the principal nilpotent operator (i.e. has one Jordan block with eigenvalue 0), then $\text{Hess}(N, h)$ is called a **regular nilpotent Hessenberg variety**. If N is nilpotent and h is the identity function $h(i) = i$ for all $1 \leq i \leq n$ then $\text{Hess}(N, h)$ is called a **nilpotent Springer variety** and often denoted \mathcal{S}_N . In this manuscript we study in some detail the regular nilpotent case, and as such sometimes notate $\text{Hess}(N, h)$ as $\text{Hess}(h)$ when N is understood to be the standard principal nilpotent operator.

Suppose given N a nilpotent matrix in standard Jordan canonical form. It turns out that for many of our statements below we must use a choice of conjugate of N which is in **highest form** [17, Definition 4.2]. We recall the following.

Definition 2.1. ([17, Definition 4.1 and Definition 4.2])

- Let X be any $m \times n$ matrix. We call the entry X_{ik} a **pivot** of X if X_{ik} is nonzero and if all entries below and to its left vanish, i.e., $X_{ij} = 0$ if $j < k$ and $X_{jk} = 0$ if $j > i$. Moreover, given i , define r_i to be the row of $X_{r_i, i}$ if the entry is a pivot, and 0 otherwise.
- Let N be an upper-triangular nilpotent $n \times n$ matrix. Then we say N is in **highest form** if its pivots form a nondecreasing sequence, namely $r_1 \leq r_2 \leq \dots \leq r_n$.

We do not require the details of the theory of highest forms of linear operators; for the purposes of the present manuscript it suffices to remark firstly that when N is the principal nilpotent matrix then N is already in highest form, and secondly that any nilpotent matrix can be conjugated by an appropriate $n \times n$ permutation matrix σ so that $N_{hf} := \sigma N \sigma^{-1}$ is in highest form. However the following observation will be relevant in Section 2.3.

Remark 2.2. *In this manuscript we always assume that our highest form $N_{hf} = \sigma N \sigma^{-1}$ has been chosen in accordance to the recipe described by Tymoczko in [17, Section 4]. Since the precise method of this construction is not relevant for the rest of the present manuscript we omit further explanation here. In the case when N is principal nilpotent we take $N_{hf} = N$ since N is already in highest form and this is the form chosen by Tymoczko in [17]. A more detailed discussion of highest forms as it pertains to poset pinball theory is in [7].*

For details on the following facts we refer the reader to e.g. [10, 11, 17] and references therein. Let N be an $n \times n$ nilpotent matrix in Jordan canonical form and let σ denote a permutation matrix such that $N_{hf} := \sigma N \sigma^{-1}$ is in highest form. It is known and straightforward to show that the following S^1 subgroup of $U(n, \mathbb{C})$ preserves $\text{Hess}(N, h)$ for N as above and any Hessenberg function h :

$$(2.2) \quad S^1 = \left\{ \begin{bmatrix} t^n & 0 & \cdots & 0 \\ 0 & t^{n-1} & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & t \end{bmatrix} \mid t \in \mathbb{C}, \|t\| = 1 \right\} \subseteq T^n \subseteq U(n, \mathbb{C}).$$

Here T^n is the standard maximal torus of $U(n, \mathbb{C})$ consisting of diagonal unitary matrices.

This implies that the conjugate circle subgroup $\sigma S^1 \sigma^{-1}$ preserves $\text{Hess}(N_{hf}, h)$. By abuse of notation we will denote both circle subgroups by S^1 , since it is clear by context which is meant. The S^1 -fixed points of $\text{Hess}(N, h)$ and $\text{Hess}(N_{hf}, h)$ are isolated, and are a subset of the T^n -fixed points of $\mathcal{F}lags(\mathbb{C}^n)$. Since the set of T^n -fixed points $\mathcal{F}lags(\mathbb{C}^n)^{T^n}$ may be identified with the Weyl group $W = S_n$, and since $\text{Hess}(N, h)^{S^1}$ (respectively $\text{Hess}(N_{hf}, h)^{S^1}$) is a subset of $\mathcal{F}lags(\mathbb{C}^n)^{T^n}$, any Hessenberg fixed point may be thought of as a permutation $w \in S_n$.

2.2. Permissible fillings, dimension pairs, lists of top parts, and associated permutations. Recall that there is a bijective correspondence between the set of conjugacy classes of nilpotent $n \times n$ complex matrices N and Young diagrams² with n boxes, given by associating to N the Young diagram λ with row lengths the sizes of the Jordan blocks of N listed in weakly decreasing order. We will use this bijection to often treat such N and λ as the same data; we sometimes denote by λ_N the Young diagram given as above corresponding to a nilpotent N .

For more details on the following see [13].

Definition 2.3. Let λ be a Young diagram with n boxes. Let $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a Hessenberg function. A **filling** of λ by the alphabet $\{1, 2, \dots, n\}$ is an injective placing of the integers $\{1, 2, \dots, n\}$ into the boxes of λ . A filling of λ is called a **(h, λ)-permissible filling** if for every horizontal adjacency $\boxed{k} \boxed{j}$ in the filling we have $k \leq h(j)$.

Remark 2.4. *In this manuscript the λ and h will frequently be understood by context. When there is no danger of confusion we simply refer to **permissible fillings**.*

Example 2.5. Let $n = 5$. Suppose $\lambda = (5)$ and $h = (3, 3, 4, 5, 5)$. Then $\boxed{2} \boxed{4} \boxed{3} \boxed{1} \boxed{5}$ is a permissible filling, whereas $\boxed{2} \boxed{3} \boxed{4} \boxed{1} \boxed{5}$ is not, since $4 \not\leq h(1)$.

We denote a permissible filling of λ by T , in analogy with standard notation for Young tableaux. Next we focus attention on certain pairs of entries in a permissible filling T .

²We use English notation for Young diagrams.

Definition 2.6. Let $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a Hessenberg function and λ a Young diagram with n boxes. A pair (a, b) is a **dimension pair** of an (h, λ) -permissible filling T of λ if the following conditions hold:

- (1) $b > a$,
- (2) b is either
 - below a in the same column of a , or
 - anywhere in a column strictly to the left of the column of a ,
 and
- (3) if there exists a box with filling c directly adjacent to the right of a , then $b \leq h(c)$.

For a dimension pair (a, b) of T , we will refer to b as the **top part** of the dimension pair.

Example 2.7. Let λ, h be as in Example 2.5. The dimension pairs in the permissible filling $\begin{bmatrix} 2 & 4 & 3 & 1 & 5 \end{bmatrix}$ are $(1, 2)$, $(1, 3)$, and $(1, 4)$. Note that $(3, 4)$ is not a dimension pair because 1 is directly to the right of the 3 and $4 \not\leq h(1)$.

Given a permissible filling T of λ , we follow [13] and denote by DP^T the set of dimension pairs of T . For each integer ℓ with $2 \leq \ell \leq n$, let

$$(2.3) \quad x_\ell := |\{(a, \ell) \mid (a, \ell) \in DP^T\}|$$

so x_ℓ is the number of times ℓ occurs as a top part in the set of dimension pairs of T . From the definitions it follows that $0 \leq x_\ell \leq \ell - 1$ for all $2 \leq \ell \leq n$. We call the integral vector $\mathbf{x} = (x_2, x_3, \dots, x_n)$ the **list of top parts** of T .

To each such \mathbf{x} we associate a permutation in S_n as follows. As a preliminary step, for each ℓ with $2 \leq \ell \leq n$ define

$$u_\ell(\mathbf{x}) := \begin{cases} s_{\ell-1}s_{\ell-2} \cdots s_{\ell-x_\ell} & \text{if } x_\ell > 0 \\ 1 & \text{if } x_\ell = 0 \end{cases}$$

where s_i denotes the simple transposition $(i, i+1)$ in S_n and 1 denotes the identity permutation. Now define the association

$$(2.4) \quad \mathbf{x} \mapsto \omega(\mathbf{x}) := u_2(\mathbf{x})u_3(\mathbf{x}) \cdots u_n(\mathbf{x}) \in S_n.$$

It is not difficult to see that (2.4) is a bijection between the set of integral vectors $\mathbf{x} \in \mathbb{Z}_{\geq 0}^{n-1}$ satisfying $0 \leq x_\ell \leq \ell - 1$ for all $2 \leq \ell \leq n - 1$ and the group S_n . In fact the word given by (2.4) is a reduced word decomposition of $\omega(\mathbf{x})$ and the x_ℓ count the number of inversions in $\omega(\mathbf{x})$ with ℓ as the higher integer. The following simple fact will be used later.

Fact 2.8. Suppose $\mathbf{x} = (x_2, \dots, x_n), \mathbf{y} = (y_2, \dots, y_n) \in \mathbb{Z}_{\geq 0}^{n-1}$ are both lists of top parts. Suppose further that for all $2 \leq \ell \leq n$, we have $x_\ell \leq y_\ell$. Then $\omega(\mathbf{x}) \leq \omega(\mathbf{y})$ in Bruhat order. This follows immediately from the definition (2.4).

Example 2.9. Continuing with Examples 2.5 and 2.7, for the permissible filling $\begin{bmatrix} 2 & 4 & 3 & 1 & 5 \end{bmatrix}$ the set DP^T of top parts of dimension pairs is $\{2, 3, 4\}$, yielding the integer vector $\mathbf{x} = (1, 1, 1, 0)$. The associated permutation $\omega(\mathbf{x})$ is then $s_1 s_2 s_3$.

Example 2.10. Let λ, h be as in Example 2.5. The filling $\begin{bmatrix} 4 & 3 & 2 & 1 & 5 \end{bmatrix}$ is also permissible, with dimension pairs $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$. Hence $\mathbf{x} = (1, 2, 1, 0)$ and the associated permutation $\omega(\mathbf{x})$ is $s_1(s_2 s_1)s_3$.

2.3. Bijection between fixed points and permissible fillings. For nilpotent Hessenberg varieties, the S^1 -fixed points $\text{Hess}(N, h)^{S^1}$ are in bijective correspondence with the set of permissible fillings of the Young diagram $\lambda = \lambda_N$, as we now describe. We will use this correspondence in the formulation of our dimension pair algorithm.

Suppose λ is a Young diagram with n boxes. We begin by defining a bijective correspondence between the set $\text{Fill}(\lambda)$ of all fillings (not necessarily permissible) of λ with permutations in S_n . Given a filling, read the entries of the filling by reading along each column from the bottom to the top, starting with the leftmost column and proceeding to the rightmost column. The association $\text{Fill}(\lambda) \leftrightarrow S_n$ is then given by interpreting the resulting word as the one-line notation of a permutation. For example the filling

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 \\ 6 \end{bmatrix}$$

has associated permutation 641523. It is easily seen that this is a bijective correspondence. Given a filling T of λ we denote its associated permutation by $\phi_\lambda(T)$.

Remark 2.11. In the case when N is the principal nilpotent $n \times n$ matrix, the corresponding Young diagram $\lambda = \lambda_N = (n)$ has only one row, so the above correspondence simply reads off the (one row of the) filling from left to right. In this case we abuse notation and denote $\phi_\lambda^{-1}(w)$ by just w . For instance, the permissible filling of $\lambda = (5)$ in Example 2.10 has associated permutation 43215.

Now let

$$(2.5) \quad \mathcal{PFill}(\lambda, h)$$

denote the set of (h, λ) -permissible fillings of λ . Recall that elements in $\text{Hess}(N, h)^{S^1}$ are viewed as permutations in S_n via the identification $\text{Flags}(\mathbb{C}^n)^{T^n} \cong S_n$. The next proposition follows from the definitions and some linear algebra. It is proven and discussed in more detail in [7], where the notation used is slightly different.

Proposition 2.12. Fix n a positive integer. Let $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a Hessenberg function and λ a Young diagram with n boxes. Suppose $N_{h,f}$ is a nilpotent operator in highest form as chosen in [17] (cf. Remark 2.2) with $\lambda_{N_{h,f}} = \lambda$. Let $\text{Hess}(N_{h,f}, h)$ denote the associated nilpotent Hessenberg variety. Then the map from the S^1 -fixed points $\text{Hess}(N_{h,f}, h)^{S^1}$ to the set of permissible fillings $\mathcal{PFill}(\lambda, h)$

$$(2.6) \quad w \in \text{Hess}(h)^{S^1} \subseteq S_n \mapsto \phi_\lambda^{-1}(w^{-1}) \in \mathcal{PFill}(\lambda, h)$$

is well-defined and is a bijection.

Remark 2.13. In the case when N is the principal nilpotent $n \times n$ matrix, λ is the Young diagram with only one row. Thus the map (2.6) above simplifies to $w \mapsto w^{-1}$ where we abuse notation (cf. Remark 2.11) and denote $\phi_\lambda^{-1}(w^{-1})$ by w^{-1} .

3. THE DIMENSION PAIR ALGORITHM FOR BETTI POSET PINBALL FOR NILPOTENT HESSENBERG VARIETIES

In this section we first explain the **dimension pair algorithm** which associates to any Hessenberg fixed point a permutation in S_n . The name is due to the fact that the construction proceeds by computing dimension pairs in appropriate permissible fillings. We then interpret this algorithm as a method for choosing **rolldowns** associated to the Hessenberg fixed points in a game of **Betti poset pinball** in the sense of [10]. The algorithm makes sense for any nilpotent Hessenberg variety, so it is defined in that generality in Section 3.1. However, our proof that the algorithm produces a successful outcome of Betti poset pinball in the sense of [10] is only for the special cases of regular nilpotent Hessenberg varieties and nilpotent Springer varieties. In Section 3.2 we briefly recall the setup and necessary results of poset pinball which allow us to conclude that our poset pinball result yields an explicit module basis for equivariant cohomology.

3.1. The dimension pair algorithm for nilpotent Hessenberg varieties. Let $N_{h,f}$ be a nilpotent $n \times n$ matrix in highest form chosen as in Remark 2.2 and let $\lambda := \lambda_{N_{h,f}}$. Let $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a Hessenberg function and $\text{Hess}(N_{h,f}, h)$ the corresponding nilpotent Hessenberg variety.

The definition of the dimension pair algorithm is pure combinatorics. It produces for each Hessenberg fixed point $w \in \text{Hess}(N_{h,f}, h)^{S^1}$ an element in S_n . Following terminology of poset pinball, we denote this function by

$$\text{roll} : \text{Hess}(N_{h,f}, h)^{S^1} \rightarrow S_n.$$

Definition 3.1. (“The dimension pair algorithm”) We define $\text{roll} : \text{Hess}(N_{h,f}, h)^{S^1} \rightarrow S_n$ as follows:

- (1) Let $w \in \text{Hess}(N_{h,f}, h)^{S^1}$ and let $\phi_\lambda^{-1}(w^{-1})$ be its corresponding permissible filling as defined in (2.6).
- (2) Let $DP^{\phi_\lambda^{-1}(w^{-1})}$ be the set of dimension pairs in the permissible filling $\phi_\lambda^{-1}(w^{-1})$.
- (3) For each ℓ with $2 \leq \ell \leq n$, set

$$x_\ell := |\{(a, \ell) \mid (a, \ell) \in DP^{\phi_\lambda^{-1}(w^{-1})}\}|$$

as in (2.3) and define $\mathbf{x} := (x_2, \dots, x_n)$.

- (4) Define $\text{roll}(w) := (\omega(\mathbf{x}))^{-1}$ where $\omega(\mathbf{x})$ is the permutation associated to the integer vector \mathbf{x} defined in (2.4).

Example 3.2. Let λ, h be as in Example 2.5. The permutation $w = 43215 \in S_n$ is in $\text{Hess}(N_{hf}, h)^{S^1}$, as can be checked. The associated permissible filling is $\begin{bmatrix} 4 & 3 & 2 & 1 & 5 \end{bmatrix}$. In Example 2.10 we saw that the associated permutation is $s_1(s_2s_1)s_3$, so we conclude $\text{roll}(w) = s_3(s_1s_2)s_1$.

We next show that the rolldown function $\text{roll} : \text{Hess}(h)^{S^1} \rightarrow S_n$ defined by the dimension pair algorithm above satisfies the conditions to be a **successful outcome of Betti poset pinball** as in [10] in certain cases of nilpotent Hessenberg varieties. The statement of one of the conditions requires advance knowledge of the Betti numbers of nilpotent Hessenberg varieties, for which we recall the following result (reformulated in our language) from [17].

Theorem 3.3. ([17, Theorem 1.1]) Let $N_{hf} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a nilpotent matrix in highest form chosen as in Remark 2.2 and let $\lambda := \lambda_{N_{hf}}$. Let $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a Hessenberg function and let $\text{Hess}(N_{hf}, h)$ denote the corresponding nilpotent Hessenberg variety. There is a paving by (complex) affine cells of $\text{Hess}(N_{hf}, h)$ such that:

- the affine cells are in one-to-one correspondence with $\text{Hess}(N_{hf}, h)^{S^1}$, and
- the (complex) dimension of the affine cell C_w corresponding to a fixed point $w \in \text{Hess}(N, h)^{S^1}$ is

$$(3.1) \quad \dim_{\mathbb{C}}(C_w) = |DP\phi_{\lambda}^{-1}(w^{-1})|.$$

In particular, Theorem 3.3 implies that the odd Betti numbers of $\text{Hess}(N_{hf}, h)$ are 0, and the $2k$ -th even Betti number is precisely the number of fixed points w in $\text{Hess}(N_{hf}, h)^{S^1}$ such that $|DP\phi_{\lambda}^{-1}(w^{-1})| = k$. Given the regular nilpotent Hessenberg variety $\text{Hess}(N_{hf}, h)$, denote by b_k its $2k$ -th Betti number, i.e.

$$b_k := \dim_{\mathbb{C}} H^{2k}(\text{Hess}(N_{hf}, h)).$$

We may now formulate the conditions that guarantee that $\text{roll} : \text{Hess}(N_{hf}, h)^{S^1} \rightarrow S_n$ is a successful outcome of Betti pinball. For more details we refer the reader to [10, Section 3]. It suffices to check the following:

- (1) $\text{roll} : \text{Hess}(N_{hf}, h)^{S^1} \rightarrow S_n$ is injective,
- (2) for every $w \in \text{Hess}(N_{hf}, h)^{S^1}$, we have $\text{roll}(w) \leq w$ in Bruhat order, and
- (3) for every $k \geq 0, k \in \mathbb{Z}$, we have

$$b_k = \left| \left\{ \text{roll}(w) \mid w \in \text{Hess}(N_{hf}, h)^{S^1} \text{ with } \ell(\text{roll}(w)) = k \right\} \right|$$

where $\ell(\text{roll}(w))$ denotes the Bruhat length of $\text{roll}(w) \in S_n$.

We prove each claim in turn. For the first assertion we restrict to two special cases of Hessenberg varieties.

Lemma 3.4. Suppose that $\text{Hess}(N_{hf}, h)$ is either a regular nilpotent Hessenberg variety or a nilpotent Springer variety. Then the function $\text{roll} : \text{Hess}(N_{hf}, h)^{S^1} \rightarrow S_n$ is injective.

Proof. Since the association $\mathbf{x} = (x_2, x_3, \dots, x_n) \mapsto \omega(\mathbf{x})$ given in (2.4) is a bijection it suffices to show that the map which sends a Hessenberg fixed point $w \in \text{Hess}(h)^{S^1}$ to the list of top parts \mathbf{x} of its associated permissible filling is injective. Mbirika shows that, in the cases of regular nilpotent Hessenberg varieties and nilpotent Springer varieties, there exists an inverse to this map (Mbirika works with monomials in $n - 1$ variables constructed from the list of top parts, but this is equivalent data) [13, Section 3.2]. The result follows. \square

Lemma 3.5. For every $w \in \text{Hess}(h)^{S^1}$, we have $\text{roll}(w) \leq w$ in Bruhat order.

Proof. Since Bruhat order is preserved under taking inverses, it suffices to prove that $\omega(\mathbf{x})$ is Bruhat-less than w^{-1} . For any permutation $u \in S_n$, set

$$y_{\ell} := \{(a, \ell) \mid (a, \ell) \text{ is an inversion in } u\}$$

and let $\mathbf{y} := (y_2, y_3, \dots, y_n)$. Then the association (2.4) applied to the vector \mathbf{y} recovers the permutation u . By definition of ϕ_{λ} and the definition of dimension pairs, the set $DP\phi_{\lambda}^{-1}(w^{-1})$ is always a subset of the set of inversions of the permutation w^{-1} . From Fact 2.8 it follows that the permutation $\omega(\mathbf{x})$ is Bruhat-less than w^{-1} as desired. \square

Lemma 3.6. *Let $N_{hf} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a nilpotent matrix in highest form chosen as in Remark 2.2 and let $\lambda := \lambda_{N_{hf}}$. Let $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a Hessenberg function and $\text{Hess}(N_{hf}, h)$ the associated nilpotent Hessenberg variety. For every $k \geq 0, k \in \mathbb{Z}$, we have*

$$b_k = \left| \left\{ \text{roll}(w) \mid w \in \text{Hess}(h)^{S^1} \text{ with } \ell(\text{roll}(w)) = k \right\} \right|,$$

where $\ell(\text{roll}(w))$ denotes the Bruhat length of $\text{roll}(w) \in S_n$.

Proof. By construction, $\text{roll}(w)$ has a reduced word decomposition consisting of precisely $|DP^{\phi_\lambda^{-1}(w^{-1})}|$ simple transpositions. Hence its Bruhat length is $|DP^{\phi_\lambda^{-1}(w^{-1})}|$. By Theorem 3.3, b_k is precisely the number of fixed points w with $|DP^{\phi_\lambda^{-1}(w^{-1})}| = k$ so the result follows. \square

The following is immediate from the above lemmas and the definition of Betti pinball given in [10, Section 3].

Proposition 3.7. *Suppose that $\text{Hess}(N_{hf}, h)$ is either a regular nilpotent Hessenberg variety or a nilpotent Springer variety. Then the association $w \mapsto \text{roll}(w)$ given by the dimension pair algorithm is a possible outcome of a successful game of Betti poset pinball played with ambient partially ordered set S_n equipped with Bruhat order, rank function $\rho = \ell : S_n \rightarrow \mathbb{Z}$ given by Bruhat length, initial subset $\text{Hess}(h)^{S^1} \subseteq S_n$, and target Betti numbers $b_k := \dim_{\mathbb{C}} H^{2k}(\text{Hess}(h); \mathbb{C})$.*

Remark 3.8. *Lemmas 3.5 and 3.6 hold for general nilpotent N_{hf} and Hessenberg functions h . Hence to prove that Proposition 3.7 holds for more general cases of nilpotent Hessenberg varieties, it suffices to check that the injectivity assertion (1) above holds. We do not know counterexamples where the injectivity fails. It would be of interest to clarify the situation for more general N_{hf} and h .*

3.2. Betti pinball, poset-upper-triangularity, and module bases. In the context of a **GKM-compatible subspace of a GKM space** [10, Definition 4.5], it is explained in [10, Section 4] that the outcome of a game of poset pinball may be interpreted as specifying a set of equivariant cohomology classes which, under additional conditions, yields a module basis for the equivariant cohomology of the GKM-compatible subspace. In this paper, the GKM space is the flag variety $\text{Flags}(\mathbb{C}^n)$ with the standard T^n -action and the GKM-compatible subspace is $\text{Hess}(N_{hf}, h)$ with the S^1 -action specified above. Consider the $H_{T^n}^*(\text{pt})$ -module basis for $H_{T^n}^*(\text{Flags}(\mathbb{C}^n))$ given by the equivariant Schubert classes $\{\sigma_w\}_{w \in S_n}$. The dimension pair algorithm then specifies the set

$$\{p_{\text{roll}(w)} \mid w \in \text{Hess}(N_{hf}, h)^{S^1}\} \subseteq H_{S^1}^*(\text{Hess}(N_{hf}, h))$$

where for any $u \in S_n$ the class $p_u := \pi(\sigma_u)$ is defined to be the image of σ_u under the natural projection map

$$\pi : H_{T^n}^*(\text{Flags}(\mathbb{C}^n)) \rightarrow H_{S^1}^*(\text{Hess}(N_{hf}, h))$$

induced by the inclusion of groups $S^1 \hookrightarrow T^n$ and the S^1 -equivariant inclusion of spaces $\text{Hess}(N_{hf}, h) \hookrightarrow \text{Flags}(\mathbb{C}^n)$. We refer to the images p_u as **Hessenberg Schubert classes**.

Following the methods of [10] we view $H_{T^n}^*(\text{Flags}(\mathbb{C}^n))$ and $H_{S^1}^*(\text{Hess}(N_{hf}, h))$ as subrings of

$$H_{T^n}^*((\text{Flags}(\mathbb{C}^n))^{T^n}) \cong \bigoplus_{w \in S_n} H_{T^n}^*(\text{pt}) \quad \text{respectively} \quad H_{S^1}^*((\text{Hess}(N_{hf}, h))^{S^1}) \cong \bigoplus_{w \in \text{Hess}(N_{hf}, h)^{S^1}} H_{S^1}^*(\text{pt}).$$

We denote by $\sigma_w(w'), p_{\text{roll}(w)}(w')$ the value of the w' -th coordinate in the direct sums above, for $w, w' \in S_n$ or $w, w' \in \text{Hess}(N_{hf}, h)^{S^1}$ respectively. If

$$(3.2) \quad p_{\text{roll}(w)}(w) \neq 0, \quad \text{and} \quad p_{\text{roll}(w)}(w') = 0 \text{ if } w \not\leq w'$$

for all $w, w' \in \text{Hess}(N_{hf}, h)^{S^1}$ then the set $\{p_{\text{roll}(w)} \mid w \in \text{Hess}(N_{hf}, h)^{S^1}\}$ in $H_{S^1}^*(\text{Hess}(N_{hf}, h))$ is called **poset-upper-triangular** (with respect to the partial order on $\text{Hess}(N_{hf}, h)^{S^1} \subseteq S_n$ induced from Bruhat order) [10, Definition 2.3]. Finally, recall that the cohomology degree of an equivariant Schubert class σ_w (and hence also the corresponding Hessenberg Schubert class p_w) is $2 \cdot \ell(w)$.

The following is immediate from [10, Proposition 4.14] and the above discussion.

Proposition 3.9. *Let $\text{Hess}(N_{hf}, h)$ be either a regular nilpotent Hessenberg variety or a nilpotent Springer variety. Let $\text{roll} : \text{Hess}(N_{hf}, h)^{S^1} \rightarrow S_n$ be the dimension pair algorithm defined above. Suppose (3.2) holds for all $w \in \text{Hess}(N_{hf}, h)^{S^1}$. Then the Hessenberg Schubert classes $\{p_{\text{roll}(w)} \mid w \in \text{Hess}(N_{hf}, h)^{S^1}\}$ form a $H_{S^1}^*(\text{pt})$ -module basis for the S^1 -equivariant cohomology ring $H_{S^1}^*(\text{Hess}(N_{hf}, h))$.*

Therefore, in order to prove that the Hessenberg Schubert classes above form a module basis as desired, it suffices to show that they satisfy the upper-triangularity conditions (3.2) for all $w, w' \in \text{Hess}(N_{hf}, h)^{S^1}$. The proof of this assertion, for a special class of regular nilpotent Hessenberg varieties closely related to Peterson varieties, is the content of Sections 4 and 5.

We close the section with a brief discussion of matchings. Following [10, Section 4.3], define

$$\deg_{\text{Hess}(N_{hf}, h)}(w) := \dim_{\mathbb{C}}(C_w)$$

to be the (complex) dimension of the affine cell C_w containing the fixed point w in Tymoczko's paving by affines of $\text{Hess}(N_{hf}, h)$ in Theorem 3.3. Then from the discussion above we know

$$\deg_{\text{Hess}(N_{hf}, h)}(w) = |DP^{\phi_{\lambda^{-1}}(w^{-1})}| = \ell(\text{roll}(w)),$$

and since the cohomology degree of $p_{\text{roll}(w)}$ is $2 \cdot \ell(\text{roll}(w))$, we see that the association $w \mapsto \text{roll}(w)$ from $\text{Hess}(N_{hf}, h)^{S^1} \rightarrow S_n$ is also a **matching** in the sense of [10] with respect to $\deg_{\text{Hess}(N_{hf}, h)}$ and rank function ρ on S_n given by Bruhat length. Thus the fact that the $\{p_{\text{roll}(w)} \mid w \in \text{Hess}(N_{hf}, h)^{S^1}\}$ form a module basis can also be deduced from [10, Theorem 4.18].

4. POSET-UPPER-TRIANGULARITY OF ROLLDOWN CLASSES FOR 334-TYPE HESSENBERG VARIETIES

In this section and in Section 5 we analyze in detail the dimension pair algorithm in the case of a Hessenberg variety which is closely related to the Peterson variety, and in particular prove that the algorithm produces a poset-upper-triangular module basis for its S^1 -equivariant cohomology ring. Here and below the nilpotent operator N under consideration is always the principal nilpotent, so we omit the N from the notation and write $\text{Hess}(h)$. Similarly the corresponding Young diagram is always $\lambda = (n)$ so we omit the λ from notation and write $\mathcal{P}\text{Fill}(h)$ instead of $\mathcal{P}\text{Fill}(\lambda, h)$.

We fix for this discussion the Hessenberg function given by

$$(4.1) \quad h(1) = h(2) = 3, \quad h(i) = i + 1 \text{ for } 3 \leq i \leq n - 1, \quad \text{and } h(n) = n.$$

The only difference between this function h and the Hessenberg function for the Peterson variety studied in [11] is that the value of $h(1)$ is 3 instead of 2. In this sense this h is “close” to the Peterson case. Thus it is natural that much of our analysis follows that for Peterson varieties in [11], although it is still necessary to introduce new ideas and terminology to handle the Hessenberg fixed points in $\text{Hess}(h)^{S^1}$ which do not arise in the Peterson case.

The Hessenberg function h in (4.1) is trivial if $n = 3$ since in that case $h(1) = h(2) = h(3) = 3$ which implies that the corresponding Hessenberg variety $\text{Hess}(h)$ is equal to the full flag variety $\text{Flags}(\mathbb{C}^3)$. Hence we assume $n \geq 4$ throughout. Under this assumption and following the notation introduced in Section 2, the Hessenberg function is of the form $h = (3, 3, 4, \dots)$. As such, for the purposes of this manuscript, we refer to this family of regular nilpotent Hessenberg varieties as **334-type Hessenberg varieties**.

Our main result is the following theorem.

Theorem 4.1. *Let $n \geq 4$ and let $\text{Hess}(h)$ be the 334-type Hessenberg variety in $\text{Flags}(\mathbb{C}^n)$. Let $\text{roll} : \text{Hess}(h)^{S^1} \rightarrow S_n$ be the dimension pair algorithm defined in Section 3. Then*

$$(4.2) \quad p_{\text{roll}(w)}(w) \neq 0, \quad \text{and} \quad p_{\text{roll}(w)}(w') = 0 \text{ if } w \not\leq w'$$

for all $w, w' \in \text{Hess}(h)^{S^1}$. In particular the Hessenberg Schubert classes $\{p_{\text{roll}(w)} \mid w \in \text{Hess}(h)^{S^1}\}$ form a $H_{S^1}^(\text{pt})$ -module basis for the S^1 -equivariant cohomology ring $H_{S^1}^*(\text{Hess}(h))$.*

For ease of exposition, and because the arguments required are of a somewhat different nature, we prove Theorem 4.1 by proving the two assertions in (4.2) separately, as follows.

Proposition 4.2. *Let $n, h, \text{Hess}(h)$ and roll be as above. Then*

$$(4.3) \quad p_{\text{roll}(w)}(w) \neq 0$$

for all $w \in \text{Hess}(h)^{S^1}$.

Proposition 4.3. *Let $n, h, \text{Hess}(h)$ and roll be as above. Then*

$$(4.4) \quad p_{\text{roll}(w)}(w') = 0 \text{ if } w \not\leq w'$$

for all $w, w' \in \text{Hess}(h)^{S^1}$.

The proof of Proposition 4.2 is the content of Section 5. The main result of the present section is the upper-triangularity property asserted in Proposition 4.3. Its proof requires a number of preliminary results. We first begin by reformulating the problem in terms of Bruhat relations among the fixed points.

Lemma 4.4. *Let $n, h, \text{Hess}(h)$ and roll be as above. If for all $w, w' \in \text{Hess}(h)^{S^1}$ we have*

$$(4.5) \quad \text{roll}(w) \leq w' \Leftrightarrow w \leq w'$$

in Bruhat order, then the Hessenberg Schubert classes $\{p_{\text{roll}(w)} \mid w \in \text{Hess}(h)^{S^1}\}$ satisfy (4.4).

Proof. Recall that the equivariant Schubert classes are poset-upper-triangular with respect to Bruhat order on S_n . In particular, for all $w, w' \in S_n$ we have $\sigma_w(w') = 0$ if $w' \not\leq w$. Since the Hessenberg Schubert classes are images of the Schubert classes and the diagram

$$(4.6) \quad \begin{array}{ccc} H_{T^n}^*(\text{Flags}(\mathbb{C}^n)) & \hookrightarrow & H_{T^n}^*((\text{Flags}(\mathbb{C}^n))^{T^n}) \cong \bigoplus_{w \in W} H_{T^n}^*(\text{pt}) \\ \downarrow & & \downarrow \\ H_{S^1}^*(\text{Hess}(h)) & \hookrightarrow & H_{S^1}^*((\text{Hess}(h))^{S^1}) \cong \bigoplus_{w \in \text{Hess}(h)^{S^1}} H_{S^1}^*(\text{pt}) \end{array}$$

commutes, it follows that if for all $w, w' \in \text{Hess}(h)^{S^1}$, we have

$$(4.7) \quad \text{roll}(w) \leq w' \Leftrightarrow w \leq w'$$

in Bruhat order then (4.4) follows. \square

The rest of this section is devoted to the proof of (4.5), which by Lemma 4.4 then proves Proposition 4.3.

4.1. Fixed points and associated subsets for the 334-type Hessenberg variety. In this section we enumerate the fixed points in the 334-type Hessenberg variety and also associate to each fixed point in $\text{Hess}(h)^{S^1}$ a subset of $\{1, 2, \dots, n-1\}$. As we show below, the set of fixed points in the Peterson variety is a subset of the fixed points of the 334-type Hessenberg variety, so the main task is to describe the new fixed points which arise in the 334-type case. We begin with a general observation.

Lemma 4.5. *Let $n \in \mathbb{N}$ and let $h, h' : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be two Hessenberg functions. If $h(i) \geq h'(i)$ for all $i, 1 \leq i \leq n$, then*

$$\text{Hess}(h') \subseteq \text{Hess}(h).$$

The inclusion $\text{Hess}(h') \hookrightarrow \text{Hess}(h)$ is S^1 -equivariant and in particular $\text{Hess}(h')^{S^1} \subseteq \text{Hess}(h)^{S^1}$ and $\mathcal{P}\text{Fill}(h') \subseteq \mathcal{P}\text{Fill}(h)$.

Proof. Let $V_\bullet = (V_i)$ denote an element in $\text{Flags}(\mathbb{C}^n)$. By definition the regular nilpotent Hessenberg variety $\text{Hess}(h')$ associated to h' is

$$(4.8) \quad \text{Hess}(h') := \{V \in \text{Flags}(\mathbb{C}^n) \mid NV_i \subseteq V_{h'(i)}, \text{ for all } 1 \leq i \leq n\}$$

where N is the principal nilpotent operator. Since $V_i \subseteq V_{i+1}$ for all $1 \leq i \leq n-1$ by definition of flags and $V_n = \mathbb{C}^n$ for all flags, if $h'(i) \leq h(i)$ for all i then $NV_i \subseteq V_{h'(i)}$ automatically implies $NV_i \subseteq V_{h(i)}$. We conclude $\text{Hess}(h') \subseteq \text{Hess}(h)$. The S^1 -equivariance of the inclusion $\text{Hess}(h') \hookrightarrow \text{Hess}(h)$ follows from the definition of the S^1 -action of (2.2). \square

Applying Lemma 4.5 to the Hessenberg function

$$(4.9) \quad h'(i) = i + 1 \text{ for } 1 \leq i \leq n - 1, h'(n) = n$$

corresponding to the Peterson variety $\text{Hess}(h')$ and h the 334-type Hessenberg function (4.1), we conclude that all fixed points in $\text{Hess}(h')^{S^1}$ also arise as fixed points in $\text{Hess}(h)^{S^1}$. We refer to the elements of $\text{Hess}(h')^{S^1}$ (viewed as elements of $\text{Hess}(h)^{S^1}$) as **Peterson-type fixed points**. It therefore remains to describe $\text{Hess}(h)^{S^1} \setminus \text{Hess}(h')^{S^1}$. It turns out to be convenient to do this by first describing $\mathcal{PFill}(h) \setminus \mathcal{PFill}(h')$.

We first introduce some terminology. Given a permutation $w = (w(1) \ w(2) \ \cdots \ w(n))$ in one-line notation and some i, ℓ , we say that the entries $\{w(i), w(i+1), \dots, w(i+\ell)\}$ form a **decreasing staircase**, or simply a **staircase**, if $w(j+1) = w(j) - 1$ for all $i \leq j < i + \ell$. For example for $w = 4327516$, the segment 432 is a staircase, but 751, though the entries decrease, is not. We will say that a consecutive series of staircases is an **increasing sequence of staircases** (or simply **increasing staircases**) if each entry in a given staircase is smaller than any entry in any following staircase (reading from left to right). For instance, $w = 654987321$ is a sequence of staircases 654, 987, and 321, but is not an increasing sequence of staircases since the entries 4, 5, 6 are not smaller than the entries in the later staircase 321. However, $w = 321654987$ is an increasing sequence of (three) staircases 321, 654, and 987.

It is shown in [11] that the S^1 -fixed points of the Peterson variety $\text{Hess}(h')$ consist precisely of those permutations $w \in S_n$ such that the one-line notation of w is an increasing sequence of staircases. Since such w are equal to their own inverses, the permissible fillings $\mathcal{PFill}(h')$ corresponding to $\text{Hess}(h')$ are precisely those which are increasing sequences of staircases (cf. Remark 2.13). We now describe the permissible fillings $\mathcal{PFill}(h)$ which are *not* Peterson-type fillings. We use the language of h -tableau trees introduced by Mbirika; see [13, Section 3.1] for definitions. Recall from Remark 2.11 that we identify permissible fillings with permutations in S_n via one-line notation.

Lemma 4.6. *Let $n \geq 4$ and let $\text{Hess}(h)$ be the 334-type Hessenberg variety in $\text{Flags}(\mathbb{C}^n)$. Let $w \in \mathcal{PFill}(h)$ be a permissible filling for $\text{Hess}(h)$ which is not of Peterson type, i.e., $w \in \mathcal{PFill}(h) \setminus \mathcal{PFill}(h')$. Then precisely one of the following hold:*

- The one-line notation of w is of the form

$$w' \ 3 \ 1 \ 2 \ w''$$

where w' is a (possibly empty) staircase such that $w' \ 3$ is also a staircase, and w'' is an increasing sequence of staircases. We refer to these as **312-type permissible fillings**.

- The one-line notation of w is of the form

$$2 \ w' \ 3 \ 1 \ w''$$

where w' is a (possibly empty) staircase such that $w' \ 3$ is also a staircase, and w'' is an increasing sequence of staircases. We refer to these as **231-type permissible fillings**.

Moreover, any filling satisfying either of the above conditions appears in $\mathcal{PFill}(h) \setminus \mathcal{PFill}(h')$.

Proof of Lemma 4.6. For any Hessenberg function $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$, Mbirika shows in [13, Section 3.2] that the Level n fillings in an h -tableau tree are precisely the permissible fillings with respect to h . For the Peterson Hessenberg function in (4.9) Mbirika's corresponding h -tableau tree has the property that for every k with $1 \leq k \leq n - 1$ and every vertex at Level k , there are precisely 2 edges going down from that vertex to a Level $k + 1$ vertex. (This is because the corresponding *degree tuple* β [13, Definition 3.1.1] has $\beta_i = 2$ for all $1 \leq i \leq n - 1$.) In the case of the 334-type Hessenberg function, by definition the h -tableau tree also has precisely 2 edges going down from every vertex at Level k for all $k \neq 2$, $1 \leq k \leq n - 1$. However, at Level 2, each vertex has not 2 but 3 edges pointing down to a vertex at Level 3.

From [13, Section 3] (cf. in particular [13, Definition 3.1.9]) it can be seen that for the case of the Peterson Hessenberg function, the corresponding h -tableau tree at Level 2 has vertices $\bullet 2 1 \bullet$ and $1 \bullet 2 \bullet$, whereas for the 334-type Hessenberg function, the Level 2 vertices have the form $\bullet 2 \bullet 1 \bullet$ and $\bullet 1 \bullet 2 \bullet$. Here the bullets indicate the locations of the h -permissible positions available for the placement of the next index 3, in the sense of [13, Section 3] (cf. in particular [13, Lemma 3.1.8]). In particular, since we saw above that the edges going down from Level 3 onwards are identical in both the Peterson and 334-type Hessenberg case, it follows that the branches of the tree emanating downwards from the two Level 3 vertices $3 2 1 \bullet$, $2 1 3 \bullet$ (coming from $\bullet 2 \bullet 1 \bullet$) and the two vertices $1 \bullet 3 2 \bullet$, $1 2 \bullet 3 \bullet$ (coming from $\bullet 1 \bullet 2 \bullet$) are identical to the

corresponding branches in the h -tableau tree for the Peterson Hessenberg function. Hence all permissible fillings at the final Level n of these branches are of Peterson type. In contrast, the branches emanating from $2 \bullet 3 1 \bullet$ and $\bullet 3 1 2 \bullet$ do not appear in the Peterson h -tableau tree, and none of the fillings appearing at Level n in these branches can be Peterson permissible fillings since a 3 appears directly before a 1. Hence it is precisely these branches which account for the permissible fillings which are not of Peterson type. As noted above, the rest of the branch only has 2 edges going down from each vertex with h -permissible positions determined exactly as in the Peterson case. In particular, except for the exceptional 3 appearing directly to the left of a 1, the fillings must consist of decreasing staircases and all possible arrangements of decreasing staircases do appear. The result follows. \square

Example 4.7. Suppose $n = 8$. Then $w = 54312876$ is an example of a 312-type permissible filling where $w' = 54$ and $w'' = 876$. An example of a 231-type permissible filling is $w = 25431876$ where $w' = 54$ and $w'' = 876$. Neither of these are permissible with respect to the Peterson Hessenberg function h' since a 3 appears directly to the left of a 1. Nevertheless, both of these fillings are closely related to the Peterson-type permissible filling $w = 54321876$; this relationship is closely analyzed and used below.

We now give explicit descriptions of the corresponding non-Peterson-type elements in $\text{Hess}(h)^{S^1}$, obtained by taking inverses of the permissible fillings described in Lemma 4.6.

Definition 4.8. Let $w \in \text{Hess}(h)^{S^1}$. We say w is a **312-type (respectively 231-type) fixed point** if its inverse w^{-1} is a permissible filling of 312-type (respectively 231-type).

As observed above, since Peterson-type permissible fillings are equal to their own inverses, in that case there is no distinction between the fillings and their associated fixed points. For the 312 and 231-types, however, this is not the case. We record the following. The proof is a straightforward computation and is left to the reader.

Lemma 4.9. Let w be a 312-type (respectively 231-type) permissible filling. Let a_2 be the integer such that $a_2 + 1$ is the first entry (respectively second entry) in the one-line notation of w . Let w^{-1} be the corresponding 312-type (respectively 231-type) fixed point. Then:

- the one-line notation of w^{-1} is the same as that of w for all ℓ -th entries with $\ell > a_2 + 1$,
- if w is 312-type, then the first $a_2 + 1$ entries of the one-line notation of w^{-1} are

$$(4.10) \quad a_2 \ a_2 + 1 \ a_2 - 1 \ a_2 - 2 \ \cdots \ 2 \ 1$$

- if w is 231-type, then the first $a_2 + 1$ entries of the one-line notation of w^{-1} are

$$(4.11) \quad a_2 + 1 \ 1 \ a_2 \ a_2 - 1 \ \cdots \ 3 \ 2$$

In the case of the Peterson variety, there is a convenient bijective correspondence between the set of S^1 -fixed points of the Peterson variety and subsets \mathcal{A} of $\{1, 2, \dots, n-1\}$ given as follows [11, Section 2.3]. Let w be a Peterson-type fixed point. Then the corresponding subset is

$$(4.12) \quad \mathcal{A} := \{i : 1 \leq i \leq n-1 \text{ and } w(i) = w(i+1) + 1\} \subseteq \{1, 2, \dots, n-1\}.$$

In the case of the 334-type Hessenber variety, it is also useful to assign a subset of $\{1, 2, \dots, n-1\}$ to each fixed point as follows.

Definition 4.10. Let $w \in \text{Hess}(h)^{S^1}$. The **associated subset of $\{1, 2, \dots, n\}$ corresponding to w** , notated $\mathcal{A}(w)$, is defined as follows:

- Suppose w is of Peterson type. Then $\mathcal{A}(w)$ is defined to be the set \mathcal{A} in (4.12).
- Suppose w is 312-type. Consider the permutation $w' := ws_1$ (i.e. swap the a_2 and the $a_2 + 1$ in the one-line notation (4.10)). This is a fixed point of Peterson type. Define $\mathcal{A}(w) := \mathcal{A}(w')$.
- Suppose w is 231-type. Consider the permutation

$$w' = ws_2s_3 \cdots s_{a_2}$$

(i.e. move the 1 to the right of the 2 in the one-line notation (4.11)). This is a fixed point of Peterson type. Define $\mathcal{A}(w) := \mathcal{A}(w')$.

Example 4.11. Suppose $n = 8$.

- Suppose w is the Peterson-type fixed point $w = 54321876$. Then $\mathcal{A}(w) = \{1, 2, 3, 4\} \cup \{6, 7\}$. This agrees with the association $w \mapsto \mathcal{A}(w)$ used in [11].
- Suppose w is the 312-type fixed point $w = 34217658$ (corresponding to the 312-type permissible filling 43127658). Then $w' = ws_1 = 43217658$ and $\mathcal{A}(w) := \mathcal{A}(w') = \{1, 2, 3\} \cup \{5, 6\}$.
- Suppose w is the 231-type fixed point $w = 51432768$ (corresponding to the 231-type permissible filling 25431768). Then $w' = 54321768$ and $\mathcal{A}(w) := \mathcal{A}(w') = \{1, 2, 3, 4\} \cup \{6\}$.

Remark 4.12. The three fixed points $w = 54321876$, $w = 45321876$, and $w = 51432876$, which are respectively of Peterson type, 312 type, and 231-type, all have the same associated subset $\mathcal{A}(w) = \{1, 2, 3, 4\} \cup \{6, 7\}$.

It is useful to observe that the 312-type and 231-type fixed points have associated subsets that always contain 1 and 2.

Lemma 4.13. Let w be a 334-type Hessenberg fixed point. Suppose further that w is not of Peterson type. Then $\{1, 2\} \subseteq \mathcal{A}(w)$.

Proof. From the explicit descriptions of the one-line notation of the 312 type (respectively 231-type) fixed points given above, we know that the initial segment $a_2 a_2 + 1 \cdots 21$ (respectively $a_2 + 1 1 a_2 \cdots 32$) in the one-line notation is such that $a_2 \geq 2$. From Definition 4.10 it follows that the first decreasing staircase of the associated Peterson-type fixed point ws_1 (respectively $ws_2s_3 \cdots s_{a_2}$) is of length at least 3. In particular, the first staircase starts with an integer k which is ≥ 3 . The result follows. \square

As noted in Remark 4.12, the association $w \mapsto \mathcal{A}(w)$ given in Definition 4.10 is *not* one-to-one and hence in particular not a bijective correspondence. This makes our analysis more complicated than in [11], but the notion is still useful for our arguments below.

4.2. Reduced word decompositions for 334-type fixed points and rolldowns. In this section we fix particular choices of reduced word decompositions for the fixed points in $\text{Hess}(h)^{S^1}$ which we use in our arguments below. We also compute, and fix choices of reduced words for, the rolldowns $\text{roll}(w)$ of the fixed points.

The association $w \mapsto \mathcal{A}(w)$ of the previous section allows us to describe these reduced word decompositions in relation to that of the Peterson-type fixed points. Let a be a positive integer and k a non-negative integer. Recall that a reduced word decomposition of the maximal element (the full inversion) in the subgroup $S_{\{a, a+1, \dots, a+k+1\}} \subseteq S_n$ is given by

$$(4.13) \quad s_a(s_{a+1}s_a)(s_{a+2}s_{a+1}s_a) \cdots (s_{a+k}s_{a+k-1} \cdots s_{a+1}s_a).$$

For the purposes of this manuscript, we call this the *standard reduced word (decomposition)* for the maximal element. (This is different from the choice of reduced word decomposition used in [11, Section 2.3].) We denote a consecutive set of integers $\{a, a+1, \dots, a+k\}$ for a a positive and k a non-negative integer by $[a, a+k]$. We say that $[a, a+k]$ is a **maximal consecutive substring** of \mathcal{A} if $[a, a+k] \subseteq \mathcal{A}$ and neither $a-1$ nor $a+k+1$ are in \mathcal{A} . It is straightforward that any subset \mathcal{A} of $\{1, 2, \dots, n-1\}$ uniquely decomposes into a disjoint union of maximal consecutive substrings

$$(4.14) \quad \mathcal{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m].$$

For instance, for $\mathcal{A} = \{1, 2, 3, 5, 6, 9, 10, 11\}$, the decomposition is $\mathcal{A} = [1, 3] \cup [5, 6] \cup [9, 11]$. For any $[a, b]$, denote by $w_{[a, b]}$ the full inversion in the subgroup $S_{[a, b+1]}$. Then it follows from Definition 4.10 (see also [11, Section 2.3]) that the Peterson-type fixed point associated to \mathcal{A} , which we denote by $w_{\mathcal{A}}$, is the product

$$(4.15) \quad w_{\mathcal{A}} := w_{[a_1, a_2]} w_{[a_3, a_4]} w_{[a_5, a_6]} \cdots w_{[a_{m-1}, a_m]}.$$

We fix a choice of reduced word decomposition of $w_{\mathcal{A}}$ given by taking the product of the standard reduced words (4.13) for each of the full inversions $w_{[a_j, a_{j+1}]}$ appearing in (4.15). For the purposes of this manuscript we call this the **standard reduced word decomposition of a Peterson-type fixed point** $w_{\mathcal{A}}$.

Example 4.14. Let $n = 7$ and let $w = 4321765$ be a Peterson-type fixed point. Then the two decreasing staircases are 4321 and 765, the associated subset $\mathcal{A}(w)$ is $\{1, 2, 3\} \cup \{5, 6\}$ with maximal consecutive strings $[1, 3] := \{1, 2, 3\}$ and $[5, 6] := \{5, 6\}$. The standard reduced word decomposition of w is

$$(4.16) \quad w_{\{1, 2, 3\} \cup \{5, 6\}} = w_{[1, 3]} w_{[5, 6]} = s_1(s_2s_1)(s_3s_2s_1)s_5(s_6s_5).$$

We now fix a reduced word decomposition of the non-Peterson-type fixed points.

Lemma 4.15. *Let $w \in \text{Hess}(h)^{S^1}$ be a fixed point which is not of Peterson type and let $\mathcal{A}(w) = [a_1, a_2] \cup [a_3, a_4] \cup \dots \cup [a_{m-1}, a_m]$ be the associated subset with its decomposition into maximal consecutive substrings.*

- *If w is 312-type then a reduced word decomposition for w is given by*

$$(4.17) \quad s_1(s_2s_1) \cdots (s_{a_2}s_{a_2-1} \cdots s_3s_2)w_{[a_3, a_4]} \cdots w_{[a_{m-1}, a_m]}$$

and

- *if w is 231-type then a reduced word decomposition for w is given by*

$$(4.18) \quad s_2(s_3s_2) \cdots (s_{a_2-1}s_{a_2-2} \cdots s_3s_2)(s_{a_2}s_{a_2-1} \cdots s_2s_1)w_{[a_3, a_4]} \cdots w_{[a_{m-1}, a_m]}$$

where the $w_{[a_\ell, a_{\ell+1}]}$ in the above expressions are assumed to be given the reduced word decomposition described in (4.13).

Proof. For the first assertion, observe that the explicit description of the one-line notation 312-type fixed points in (4.10) implies that w has precisely 1 fewer inversion than $w_{\mathcal{A}(w)}$. An explicit computation shows that the given word (4.17) is equal to w , so it is a word decomposition of w with exactly as many simple transpositions as the Bruhat length of w . In particular it must be reduced. A similar argument proves the second assertion. \square

Example 4.16. *Suppose $n = 7$. Suppose $w = 3421765$ is a 312-type fixed point. Then the reduced word decomposition of w given in Lemma 4.15 is*

$$w = s_1(s_2s_1)(s_3s_2)s_5(s_6s_5).$$

Similarly suppose $w = 4132765$ is a 231-type fixed point. Then the reduced word decomposition of w given in Lemma 4.15 is

$$w = s_2(s_3s_2s_1)s_5(s_6s_5).$$

Henceforth we always use the reduced words given above.

Next we explicitly describe the rolldowns $\text{roll}(w)$ associated to each w in $\text{Hess}(h)^{S^1}$ by the dimension pair algorithm. We begin with the Peterson-type fixed points. It turns out there are two important subcases of Peterson-type fixed points.

Definition 4.17. We say that a Peterson-type fixed point w **contains the string 321** (or simply **contains 321**) if, in the one-line notation of w , the string 321 appears (equivalently, if $\{1, 2\} \subseteq \mathcal{A}(w)$). We say w **does not contain the string 321** (or simply **does not contain 321**) otherwise.

Remark 4.18. *Note that Definition 4.17 is different from the standard notion of pattern-containing or pattern-avoiding permutations since here we require the one-line notation of w to contain the string 321 exactly.*

Given a subset $\mathcal{A} = \{j_1 < j_2 < \dots < j_k\} \subseteq \{1, 2, \dots, n-1\}$ and corresponding Peterson-type fixed point $w_{\mathcal{A}}$, we call the permutation

$$(4.19) \quad s_{j_k}s_{j_{k-1}} \cdots s_{j_2}s_{j_1} \in S_n$$

the **Peterson case rolldown of $w_{\mathcal{A}}$** . Note that the word (4.19) is in fact a reduced word decomposition of this permutation; we always use this choice of reduced word. The terminology is motivated by the fact that (4.19) is the (inverse of the) permutation given in [11, Definition 4.1]. (The fact that it is the inverse of the permutation used in [11] does not affect the theory very much, as is explained in [11, Proposition 5.16].)

Lemma 4.19. *Let $n \geq 4$ and $\text{Hess}(h)$ the 334-type Hessenberg variety in $\text{Flags}(\mathbb{C}^n)$. Let w be a Peterson-type fixed point and let $\mathcal{A}(w) = \{j_1 < j_2 < \dots < j_k\}$ be its associated subset.*

- *Suppose w does not contain 321. Then $\text{roll}(w)$ is the Peterson case rolldown of $w_{\mathcal{A}(w)}$.*
- *Suppose w does contain 321, i.e., $\mathcal{A}(w) = \{j_1 < j_2 < \dots < j_k\}$ for $k \geq 2$ and $j_1 = 1$ and $j_2 = 2$. Then $\text{roll}(w)$ is*

$$(4.20) \quad \text{roll}(w) = s_{j_k}s_{j_{k-1}} \cdots s_{j_3}s_1s_2s_1.$$

In particular, if a Peterson-type fixed point w contains 321, then its rolldown $\text{roll}(w)$ is Bruhat-greater, and has Bruhat length 1 greater, than the Peterson case rolldown of w .

Proof. If w contains a 321, then by Definition 2.6, the pairs $(1, 3)$, $(2, 3)$ and $(1, 2)$ are all dimension pairs in w . Hence 3 appears precisely twice as a top part of a dimension pair and 2 appears precisely once. Thus by construction the dimension pair algorithm the permutation $\omega(\mathbf{x})$ begins with the word $s_1(s_2s_1)$. With respect to all other indices $j \in \mathcal{A}(w)$, the 334-type Hessenberg function is identical to the Peterson Hessenberg function and hence for each such j , the index $j + 1$ appears precisely once as a top part of a dimension pair of w and thus contributes precisely one s_j to $\omega(\mathbf{x})$. Taking the inverse yields (4.20) as desired.

If w does not contain 321, then 3 appears at most once as the top part of a dimension pair in w , and again for all other indices the computations are identical to the Peterson case as above. Hence $\text{roll}(w)$ is identical to the Peterson case rolldown. This completes the proof. \square

Next, we give an explicit description, along with a choice of reduced word decomposition, of the roll-downs corresponding to the non-Peterson-type fixed points.

Lemma 4.20. *Let $w \in \text{Hess}(h)^{S^1}$ and suppose that w is not of Peterson type. Let $\mathcal{A}(w) = \{j_1 = 1 < j_2 = 2 < j_3 < \dots < j_k\}$ for some $k \geq 2$.*

(1) *If w is of 312-type, then the dimension pair algorithm associates to w the permutation*

$$(4.21) \quad \text{roll}(w) = s_{j_k} s_{j_{k-1}} \cdots s_{j_4} s_{j_3} s_1 s_2.$$

(2) *If w is 231-type, then the dimension pair algorithm associates to w the permutation*

$$(4.22) \quad \text{roll}(w) = s_{j_k} s_{j_{k-1}} \cdots s_{j_4} s_{j_3} s_2 s_1.$$

Proof. Suppose w is a 312-type fixed point so $\phi_\lambda^{-1}(w^{-1})$ is a 312-type permissible filling. By definition of dimension pairs, 2 does not appear as the top part of any dimension pair (since it appears to the right of a 1). Also by definition, 3 appears as a top part of the two dimension pairs $(1, 3)$ and $(2, 3)$. The form of the 312-type permissible fillings described in Lemma 4.6 and the definition of $\mathcal{A}(w)$ imply that the other dimension pairs are precisely the pairs $(j, j + 1)$ for $j \in \mathcal{A}(w)$ (for $j \neq 1, 2$), from which it follows that $\omega(\mathbf{x}) = s_2 s_1 s_{j_3} s_{j_4} \cdots s_{j_{k-1}} s_{j_k}$. Taking inverses yields (4.21). The proof of the second assertion is similar. \square

Example 4.21. • Suppose $w = 54321876$. This is of Peterson type. Then $\text{roll}(w) = (\omega(\mathbf{x}))^{-1} = s_7 s_6 s_4 s_3 s_1 s_2 s_1$.

• Suppose $w = 45321876$. This is 312-type. Then $\text{roll}(w) = (\omega(\mathbf{x}))^{-1} = s_7 s_6 s_4 s_3 s_1 s_2$.

• Suppose $w = 51432876$. This is 231-type. Then $\text{roll}(w) = (\omega(\mathbf{x}))^{-1} = s_7 s_6 s_4 s_3 s_2 s_1$.

We conclude the section with a computation of the one-line notation of the roll-downs for different types; we leave proofs to the reader.

Lemma 4.22. *Let w be a 334-type Hessenberg fixed point and let $\mathcal{A}(w) = [a_1, a_2] \cup \dots \cup [a_{m-1}, a_m]$ be its associated subset with its decomposition into maximal consecutive substrings. Suppose w is of Peterson type that contains 321, 312-type, or of 231 type. Then $a_1 = 1$, $a_2 \geq 2$ and the first $a_2 + 1$ entries of the one-line notation of $\text{roll}(w)$ is*

$$(4.23) \quad a_2 + 1 \ 2 \ 1 \ 3 \ 4 \ \cdots \ a_2$$

for w of Peterson type that contains 321,

$$(4.24) \quad 2 \ a_2 + 1 \ 1 \ 3 \ 4 \ \cdots \ a_2$$

for w 312-type, and

$$(4.25) \quad a_2 + 1 \ 1 \ 2 \ 3 \ \cdots \ a_2$$

for w 231-type.

4.3. Bruhat order relations. In this section we analyze the properties of the association $w \mapsto \mathcal{A}(w)$ with respect to comparisons in Bruhat order.

The first two lemmas are straightforward and proofs left to the reader.

Lemma 4.23. *Let $\mathcal{A} \subseteq \{1, 2, \dots, n-1\}$ and let $w_{\mathcal{A}}$ be the Peterson-type filling associated to \mathcal{A} . Then $w_{\mathcal{A}}$ is maximal in the subgroup $S_{\mathcal{A}}$ of S_n generated by the simple transpositions $\{s_i\}_{i \in \mathcal{A}}$. In particular, $w_{\mathcal{A}}$ is Bruhat-bigger than any permutation $w \in S_{\mathcal{A}}$.*

Lemma 4.24. *Let $w \in \text{Hess}(h)^{S^1}$. Suppose w is not of Peterson type. Then w is Bruhat-less than the Peterson type fixed point $w_{\mathcal{A}(w)}$ corresponding to $\mathcal{A}(w)$.*

We also observe that a Bruhat relation $w < w'$ implies a containment relation of the associated subsets.

Lemma 4.25. *Let $w, w' \in \text{Hess}(h)^{S^1}$ and let $\mathcal{A}(w), \mathcal{A}(w')$ be the respective associated subsets. Let s_i be a simple transposition. Then:*

- (1) $s_i < w$ if and only if $i \in \mathcal{A}(w)$,
- (2) $s_i < \text{roll}(w)$ if and only if $i \in \mathcal{A}(w)$,
- (3) if $w \leq w'$ or $\text{roll}(w) \leq w'$ then $\mathcal{A}(w) \subseteq \mathcal{A}(w')$.

Proof. Bruhat order is independent of choice of reduced word decomposition for w . Therefore a simple transposition s_i is less than w in Bruhat order if and only if s_i appears in a (and hence any) reduced word decomposition of w . In particular, to prove the first claim it suffices to observe that by the definitions of $\mathcal{A}(w)$, the index i appears in $\mathcal{A}(w)$ precisely when s_i appears in the choice of reduced word for w given above. A similar argument using the explicit reduced words given for $\text{roll}(w)$ in Lemmas 4.19 and 4.20 proves the second claim. The last claim follows from the first two. \square

We have just seen that $w \leq w'$ implies $\mathcal{A}(w) \subseteq \mathcal{A}(w')$. In the case of the Peterson variety $\text{Hess}(h')$ these Bruhat relations are precisely encoded by the partial ordering given by containment of the $\mathcal{A}(w)$; specifically, by Lemma 4.23, $w_A \leq w_B$ if and only if $\mathcal{A} \subseteq \mathcal{B}$. In our 334-type Hessenberg case this is no longer true, although the sets $\mathcal{A}(w)$ do still encode the Bruhat data. The precise statements occupy the next several lemmas.

We take a moment to recall the **tableau criterion** for determining Bruhat order in the Weyl group S_n (see e.g. [3]) which will be useful in the discussion below. For $w \in S_n$, denote by $D_R(w)$ the descent set of w , namely,

$$D_R(w) := \{i \mid w(i) > w(i+1), 1 \leq i \leq n-1\}.$$

For example, for $w = 368475912$ the descent set is $D_R(w) = \{3, 5, 7\}$.

Theorem 4.26. (“The tableau criterion” [3, Theorem 2.6.3]) *For $w, v \in S_n$, let $w_{i,k}$ be the i -th element in the increasing rearrangement of $w(1), w(2), \dots, w(k)$, and similarly for $v_{i,k}$. Then $w \leq v$ in Bruhat order if and only if*

$$(4.26) \quad w_{i,k} \leq v_{i,k} \text{ for all } k \in D_R(w) \text{ and } 1 \leq i \leq k.$$

For example, suppose $w = 368475912$ and $v = 694287531$. Since $D_R(w) = \{3, 5, 7\}$, we examine the three increasing rearrangements of initial segments of w and v of lengths 3, 5, and 7 respectively, which we may organize into Young tableaux:

w

3	4	5	6	7	8	9
3	4	6	7	8		
3	6	8				

v

2	4	5	6	7	8	9
2	4	6	8	9		
4	6	9				

Comparing corresponding entries, there are two violations of the tableau condition of the proposition ($3 > 2$) in the upper-left corner, so we conclude that $w \not\leq v$.

Now we observe that some Bruhat relations never arise.

Lemma 4.27. *Let $w, w' \in \text{Hess}(h)^{S^1}$. Let $\mathcal{A}(w) = [a_1, a_2] \cup \dots \cup [a_{m-1}, a_m]$ be the associated subset of w with its decomposition into maximal consecutive substrings. Suppose one of the following conditions hold:*

- (1) w' is of Peterson type that does not contain 321 while w is not,
- (2) w' is 231-type while w is either of Peterson type that contains 321 or is 312-type.

Then $w \not\leq w'$ and $\text{roll}(w) \not\leq w'$.

Proof. If w' is of Peterson type that does not contain 321, then $\{1, 2\} \not\subseteq \mathcal{A}(w')$ by definition of the associated subsets. All other types (Peterson type that contains 321, or 312-type, or 231-type) have associated subsets containing $\{1, 2\}$ by Lemma 4.13 and by definition of $\mathcal{A}(w)$. The claim (1) now follows from Lemma 4.25.

Next suppose w' is 231-type and w is of Peterson type that contains 321. Then the first two entries of the one-line notation of w must be both strictly greater than 1, and $2 \in D_R(w)$. Similarly if w is a 312-type fixed point then $a_2 \geq 2$. From (4.10) it follows that the first two entries in the one-line notation of w are also strictly greater than 1, and $2 \in D_R(w)$. On the other hand, the one-line notation for a 231-type fixed point in (4.11) has a 1 in the second entry. By the tableau criterion, if $w < w'$ then since $2 \in D_R(w)$ in both cases under consideration, we must have that one of the first two entries of w is equal to 1, but we have

just seen that is impossible. Hence $w \not\prec w'$. The assertion that $\text{roll}(w) \not\prec w'$ follows by a similar argument using (4.23), (4.24), and (4.28). \square

For the next lemma and below, we say two fixed points are **of the same type** if both are Peterson-type, or both are 312-type, or both are 231-type.

Lemma 4.28. *Let $w, w' \in \text{Hess}(h)^{S^1}$. Suppose one of the following conditions hold:*

- *w and w' are of the same type, or*
- *w is of Peterson type and does not contain 321, and w' is either 312-type or 231-type, or*
- *w is either 312-type or 231-type, and w' is of Peterson type.*

Then

$$w < w' \text{ if and only if } \mathcal{A}(w) \subseteq \mathcal{A}(w').$$

Proof. Since the lemma above shows that $w < w'$ implies $\mathcal{A}(w) \subseteq \mathcal{A}(w')$, for all cases it suffices to show the reverse implication. First suppose w and w' are of the same type and $\mathcal{A}(w) \subseteq \mathcal{A}(w')$. An examination of the reduced word decompositions of the 334-type fillings given in the above discussion and an argument similar to that in [11] implies $w < w'$. Now suppose w is of Peterson type and does not contain 321 and w' is either of 312 type or 231-type. Then since $\{1, 2\} \not\subseteq \mathcal{A}(w)$, either $1 \notin \mathcal{A}(w)$ or $2 \notin \mathcal{A}(w)$. From the explicit reduced word decompositions of 312 or 231-type fixed points chosen above it can be seen that w' is Bruhat-greater than both $w_{\mathcal{A}(w') \setminus \{1\}}$ and $w_{\mathcal{A}(w') \setminus \{2\}}$. The claim now follows from Lemma 4.23. Finally suppose w is either 312-type or 231-type and w' is of Peterson type. Since $\mathcal{A}(w) \subseteq \mathcal{A}(w')$ we know from Lemma 4.23 that $w_{\mathcal{A}(w)} < w_{\mathcal{A}(w')} = w'$. Lemma 4.24 shows that $w < w_{\mathcal{A}(w)}$ so the result follows. \square

The next step is to show that Bruhat relations between certain Hessenberg fixed points are connected to lengths of initial maximal consecutive substrings in the associated subsets. We need some notation. Let $\mathcal{A} \subseteq \{1, 2, \dots, n-1\}$. Recall we denote by $w_{\mathcal{A}}$ the Peterson-type fixed point associated to \mathcal{A} . For the purposes of this discussion we let $u_{\mathcal{A}}$ (respectively $v_{\mathcal{A}}$) denote the 312-type (respectively 231-type) fixed point with associated subset \mathcal{A} . Thus for $\mathcal{A} = [1, a]$ for some a with $2 \leq a \leq n-1$, we have

$$\begin{aligned} (4.27) \quad u_{[1,a]} &= (a+1 \ a \cdots 3 \ 1 \ 2 \ a + 2 \ a + 3 \cdots n)^{-1} \\ &= a \ a + 1 \ a - 1 \ a - 2 \cdots 2 \ 1 \ a + 2 \ a + 3 \cdots n \end{aligned}$$

and

$$\begin{aligned} (4.28) \quad v_{[1,a]} &= (2 \ a + 1 \ a \cdots 4 \ 3 \ 1 \ a + 2 \ a + 3 \cdots n)^{-1} \\ &= a + 1 \ 1 \ a \ a - 1 \cdots 3 \ 2 \ a + 2 \ a + 3 \cdots n \end{aligned}$$

in one-line notation. For general subsets

$$\mathcal{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m],$$

with $a_1 = 1$ and $a_2 \geq 2$, the definitions 312-type and 231-type fixed points imply that

$$(4.29) \quad u_{\mathcal{A}} = u_{[a_1, a_2]} w_{[a_3, a_4]} \cdots w_{[a_{m-1}, a_m]}$$

and

$$(4.30) \quad v_{\mathcal{A}} = v_{[a_1, a_2]} w_{[a_3, a_4]} \cdots w_{[a_{m-1}, a_m]}.$$

Lemma 4.29. *Let \mathcal{A}, \mathcal{B} be subsets of $\{1, 2, \dots, n-1\}$ and let*

$$\mathcal{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m] \quad \text{and} \quad \mathcal{B} = [b_1, b_2] \cup [b_3, b_4] \cup \cdots \cup [b_{m-1}, b_m]$$

be the respective decompositions into maximal consecutive substrings. Assume both \mathcal{A} and \mathcal{B} contain $\{1, 2\}$. Let $w_{\mathcal{A}}$ (respectively $v_{\mathcal{A}}$) be the Peterson-type (respectively 231-type) fixed point corresponding to \mathcal{A} and let $u_{\mathcal{B}}$ be the 312-type fixed point corresponding to \mathcal{B} . Then

$$w_{\mathcal{A}} < u_{\mathcal{B}} \text{ (respectively } v_{\mathcal{A}} < u_{\mathcal{B}}) \text{ if and only if } \mathcal{A} \subseteq \mathcal{B} \text{ and } b_2 \geq a_2 + 1.$$

Proof. We begin by recalling two basic observations about Bruhat order in S_n . Both follow straightforwardly from its definition in terms of reduced word decompositions. Suppose $w, w' \in S_n$ and assume that w and w' do not share any simple transpositions in their reduced word decompositions, i.e., $s_i < w$ implies $s_i \not\prec w'$ and vice versa. Then firstly, $w \cdot w' < w''$ for $w'' \in S_n$ if and only if both $w < w''$ and $w' < w''$. Secondly, $w < w' \cdot w''$ if and only if $w < w''$.

Recall that $w_{\mathcal{A}}$ can be written as

$$(4.31) \quad w_{\mathcal{A}} = w_{[a_1, a_2]} \cdot w_{[a_3, a_4]} \cdots w_{[a_{m-1}, a_m]}.$$

Moreover each factor appearing in the decomposition (4.31) (respectively (4.30) and (4.29)) for $w_{\mathcal{A}}$ (respectively $v_{\mathcal{A}}$ and $u_{\mathcal{B}}$) has the property that it does not share any simple transpositions with any other factor appearing in the decomposition.

Now suppose $v_{\mathcal{A}}$ (respectively $w_{\mathcal{A}}$) is Bruhat-less than $u_{\mathcal{B}}$. Then we know from Lemma 4.25 that $\mathcal{A} \subseteq \mathcal{B}$ so it suffices to prove $b_2 \geq a_2 + 1$. From Lemma 4.9 and the definition of the Peterson type fixed points we know that the one-line notation for $v_{\mathcal{A}}$ (respectively $w_{\mathcal{A}}$) has first $a_2 + 1$ entries

$$a_2 + 1 \ 1 \ a_2 \ a_2 - 1 \ \cdots \ 3 \ 2$$

(respectively $a_2 + 1 \ a_2 \ a_2 - 1 \ \cdots \ 3 \ 2 \ 1$) while the one-line notation of $u_{\mathcal{B}}$ has first $b_2 + 1$ entries given by

$$b_2 \ b_2 + 1 \ b_2 - 1 \ \cdots \ 3 \ 2 \ 1.$$

In particular $1 \in D_R(v_{\mathcal{A}})$ and also $1 \in D_R(w_{\mathcal{A}})$. By the tableau criterion, this implies that the first entry of the one-line notation of $v_{\mathcal{A}}$ and $w_{\mathcal{A}}$ must be less than or equal to the first entry of that of $u_{\mathcal{B}}$. Hence $a_2 + 1 \leq b_2$ as desired.

Conversely suppose $\mathcal{A} \subseteq \mathcal{B}$ and $b_2 \geq a_2 + 1$. Then an examination of the one-line notation of $v_{[a_1, a_2]}$ (respectively $w_{[a_1, a_2]}$) compared to that of $u_{[b_1, b_2]}$ and another application of the tableau criterion implies that $v_{[a_1, a_2]} < u_{[b_1, b_2]}$ and $w_{[a_1, a_2]} < u_{[b_1, b_2]}$. In particular $v_{[a_1, a_2]}$ and $w_{[a_1, a_2]}$ are also Bruhat-less than $u_{\mathcal{B}}$. Moreover since $\mathcal{A} \subseteq \mathcal{B}$ it follows that $[a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m] \subseteq \mathcal{B}$ so Lemma 4.23 implies $w' := w_{[a_3, a_4]} \cdots w_{[a_{m-1}, a_m]} < w_{\mathcal{B}} = u_{\mathcal{B}} \cdot s_1$, where the last equality follows from Lemma 4.9. Since s_1 does not appear in any factor of w' the general fact above implies $w' < u_{\mathcal{B}}$. Finally since neither $w_{[a_1, a_2]}$ nor $v_{[a_1, a_2]}$ share any simple transpositions with w' the other general fact above yields $v_{\mathcal{A}} < u_{\mathcal{B}}$, $w_{\mathcal{A}} < u_{\mathcal{B}}$ as desired. \square

4.4. Proof of Proposition 4.3. We may now prove the upper-triangular vanishing property of 334-type Hessenberg Schubert classes.

Proof of Proposition 4.3. Let $w, w' \in \text{Hess}(h)^{S^1}$ and let

$$\mathcal{A}(w) = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m] \quad \text{and} \quad \mathcal{A}(w') = [a'_1, a'_2] \cup [a'_3, a'_4] \cup \cdots \cup [a'_{r-1}, a'_r]$$

be the respective associated subsets decomposed into maximal consecutive substrings. By Lemmas 3.5 and 4.4 it suffices to prove that if $\text{roll}(w) \leq w'$, then $w \leq w'$. So suppose $\text{roll}(w) \leq w'$. By Lemma 4.25 this implies $\mathcal{A}(w) \subseteq \mathcal{A}(w')$. By Lemma 4.28 we can conclude $w \leq w'$ if one of the following hold:

- w and w' are of the same type, or
- w is of Peterson type that does not contain 321, and w' is either 312-type or 231-type, or
- w is either 312-type or 231-type, and w' is of Peterson type.

Now suppose one of the following holds:

- w' is of Peterson type that does not contain 321 and w is not,
- w' is 231-type and w is of Peterson type that contains 321, or
- w' is 231-type and w is 312-type.

In these cases, Lemma 4.27 implies that $\text{roll}(w) \not\leq w'$ so there is nothing to prove.

It remains to discuss the cases when:

- w is of Peterson type that contains 321 and w' is 312-type, or
- w is 231-type and w' is 312-type.

By Lemma 4.29 it suffices to show that $a'_2 \geq a_2 + 1$. Suppose w is of Peterson type that contains 321. In particular $a_2 \geq 2$. From (4.23) we know that $1 \in D_R(\text{roll}(w))$ and the first entry in the one-line notation of $\text{roll}(w)$ is $a_2 + 1$. On the other hand (4.27) implies the one-line notation of w' begins with a'_2 . So if $\text{roll}(w) < w'$ then the tableau criterion implies $a'_2 \geq a_2 + 1$ as desired. Now suppose w is of 231-type. Again $a_2 \geq 2$ and from (4.25) we know $\text{roll}(w)$ has $1 \in D_R(\text{roll}(w))$ and $a_2 + 1$ as its first entry. By the same argument, $a'_2 \geq a_2 + 1$ as desired. The result follows. \square

5. COMBINATORIAL FORMULAE FOR RESTRICTIONS TO FIXED POINTS OF 334-TYPE HESSENBERG SCHUBERT CLASSES

Our goal in this section is to give a combinatorial formula for $p_{\text{roll}(w)}(w)$ from which it follows as a corollary that it is nonzero. This proves Proposition 4.2 and hence Theorem 4.1. Although not strictly necessary for the proof of Proposition 4.2 we choose to prove the explicit formula (Proposition 5.9 below) since such a formula is a first step towards a derivation of a Monk formula for 334-type Hessenberg varieties and because it conveys a flavor of the combinatorics embedded in the GKM theory of Hessenberg varieties which are larger than the Peterson varieties in [11]. Many of our computations are analogues of those in [11, Section 5]. Our main tool is **Billey's formula**. We briefly recall some definitions and results (see also discussion in [11, Section 4]).

Definition 5.1. ([11, Definition 4.7]) Given a permutation $w \in S_n$, an index $j \in \{1, 2, \dots, \ell(w)\}$, and a choice of reduced word decomposition $\mathbf{b} = (b_1, b_2, \dots, b_{\ell(w)})$ (corresponding to the word $w = s_{b_1} s_{b_2} \cdots s_{b_{\ell(w)}}$) for w , define

$$(5.1) \quad r(j, \mathbf{b}) := s_{b_1} s_{b_2} \cdots s_{b_{j-1}} (t_{b_j} - t_{b_{j+1}}).$$

From the definition it follows that $r(j, \mathbf{b})$ is an element of $H_T^*(\text{pt}) \cong \text{Sym}(\mathfrak{t}^*) \cong \mathbb{C}[t_1, t_2, \dots, t_n]$ of the form $t_\ell - t_k$ for some ℓ, k . These elements $r(j, \mathbf{b})$ are the building blocks of Billey's formula [2, Theorem 4] which computes the restrictions $\sigma_v(w)$ of equivariant Schubert classes σ_v at arbitrary permutations w in S_n .

Theorem 5.2. ("Billey's formula", [2, Theorem 4]) Let $w \in S_n$. Fix a reduced word decomposition $w = s_{b_1} s_{b_2} \cdots s_{b_{\ell(w)}}$ and let $\mathbf{b} = (b_1, b_2, \dots, b_{\ell(w)})$ be the sequence of its indices. Let $v \in S_n$. Then the restriction $\sigma_v(w)$ of the Schubert class σ_v at the T -fixed point w is given by

$$(5.2) \quad \sigma_v(w) = \sum r(j_1, \mathbf{b}) r(j_2, \mathbf{b}) \cdots r(j_{\ell(v)}, \mathbf{b})$$

where the sum is taken over subwords $s_{b_{j_1}} s_{b_{j_2}} \cdots s_{b_{j_{\ell(v)}}}$ of \mathbf{b} that are reduced words for v .

We record the following fact, used in the proof below, which follows straightforwardly from the Billey formula.

Fact 5.3. Suppose $v, w \in S_n$ with $v \leq w$ in Bruhat order. Suppose there exists a decomposition $w = w' \cdot w''$ for $w', w'' \in S_n$ where $v \leq w'$ and, for all simple transpositions s_i such that $s_i < v$, we have $s_i \not\leq w''$. Then $\sigma_v(w) = \sigma_v(w')$.

Following terminology in [11], we refer to an individual summand of the expression in the right hand side of (5.2), corresponding to a single reduced subword $v = s_{b_{j_1}} s_{b_{j_2}} \cdots s_{b_{j_{\ell(v)}}}$ of w , as a **summand in Billey's formula**. In order to derive formulas for $p_v(w)$ where p_v is a Hessenberg Schubert class, we use the linear projection $\pi_{S^1} : \mathfrak{t}^* \rightarrow \text{Lie}(S^1)^*$ dual to the inclusion of our circle subgroup S^1 into T given by (2.2). More specifically, since the diagram (4.6) commutes, we have

$$(5.3) \quad p_v(w) = \sum \pi_{S^1}(r(j_1, \mathbf{b})) \pi_{S^1}(r(j_2, \mathbf{b})) \cdots \pi_{S^1}(r(j_{\ell(v)}, \mathbf{b})).$$

We refer to the right hand side of the above equality as **Billey's formula for $p_v(w)$** . Recall $\pi_{S^1}(t_\ell - t_{k+1}) = (k+1-\ell)t$ for a positive root $t_\ell - t_{k+1}$ [11, Section 5].

We also use the following.

Definition 5.4. ([11, Definition 5.4]) Fix $\mathcal{A} \subseteq \{1, 2, \dots, n-1\}$. Define $\mathcal{H}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\mathcal{H}_{\mathcal{A}}(j) = \text{the maximal element in the maximal consecutive substring of } \mathcal{A} \text{ containing } j.$$

Definition 5.5. ([11, Definition 5.5]) Fix $\mathcal{A} \subseteq \{1, 2, \dots, n-1\}$. Define $\mathcal{T}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\mathcal{T}_{\mathcal{A}}(j) = \text{the minimal element in the maximal consecutive substring of } \mathcal{A} \text{ containing } j.$$

We proceed to some preliminary computations. Let $\mathbf{b} = (b_1, \dots, b_{\ell(w)})$ be a reduced word decomposition $w = s_{b_1} s_{b_2} \cdots s_{b_{\ell(w)}}$ of w and let i be an index appearing in \mathbf{b} , i.e. $b_\ell = i$ for some $1 \leq \ell \leq \ell(w)$. Our first computation, Lemma 5.6, gives an expression for $\pi_{S^1}(r(\ell, \mathbf{b}))$ which shows in particular that the value of $\pi_{S^1}(r(\ell, \mathbf{b}))$ depends only on the value of the index $b_\ell = i$ and not on its location ℓ in the word \mathbf{b} . Note that if $v = s_i$ then the summands in Billey's formula for $p_v(w) = p_{s_i}(w)$ are precisely equal to $r(\ell, \mathbf{b})$ for each ℓ such that $b_\ell = i$. Thus an equivalent formulation of the claim is that the summands in Billey's formula

for $p_{s_i}(w)$ are all equal. This is analogous to a result in the Peterson case [11, Lemma 5.2] except that in our situation, the form of the formulas depend on the index i as well as on the type of the fixed point w in question.

Lemma 5.6. *Let $w \in \text{Hess}(h)^{S^1}$ and let $\mathbf{b} = (b_1, \dots, b_{\ell(w)})$ be the reduced word decomposition of w chosen in Section 4. Let $\mathcal{A}(w) = [a_1, a_2] \cup [a_3, a_4] \cup \dots \cup [a_{m-1}, a_m]$ be the associated subset of w decomposed into maximal consecutive substrings. Let $i \in \{1, 2, \dots, n-1\}$.*

- (1) *If $i \notin \mathcal{A}(w)$, then each summand in Billey's formula for $p_{s_i}(w)$ is 0. In particular, $p_{s_i}(w) = 0$.*
- (2) *Suppose $i \in \mathcal{A}(w)$ and suppose one of the following conditions hold:*
 - *w is of Peterson type, or*
 - *w is 312-type, or*
 - *w is 231-type and $i \notin [a_1, a_2]$.*

Then each summand in Billey's formula for $p_{s_i}(w)$ is equal to

$$(i - \mathcal{T}_{\mathcal{A}(w)}(i) + 1)t.$$

- (3) *Suppose w is 231-type and $i = 1$. Then each summand in Billey's formula for $p_{s_i}(w)$ is equal to*

$$a_2 t = \mathcal{H}_{\mathcal{A}(w)}(1)t.$$

- (4) *Suppose w is 231-type and $i \in [2, a_2]$. Then each summand in Billey's formula for $p_{s_i}(w)$ is equal to*

$$(i - \mathcal{T}_{\mathcal{A}(w)}(i))t = (i - 1)t.$$

Proof. If i does not occur in $\mathcal{A}(w)$ then each summand is 0 by Billey's formula for $\sigma_{s_i}(w)$, since $s_i \not\prec w$ and thus never appears in the reduced word decomposition of w . For the next claim, the fact that each summand is equal to $(i - \mathcal{T}_{\mathcal{A}(w)}(i) + 1)t$ for the listed cases follows from examination of the chosen reduced word decompositions of w and an argument identical to that in [11]. Thus it remains to check the cases in which the summand differs from the case of Peterson varieties. First suppose w is 231-type and that $i = 1$. From the choice of explicit reduced word decomposition for such w given in (4.18) and Billey's formula, it follows that each summand in Billey's formula for $\sigma_{s_1}(w)$ is equal to

$$(5.4) \quad \begin{aligned} r_2(r_3r_2) \cdots (r_{a_2-1}r_{a_2-2} \cdots r_3r_2)(r_{a_2}r_{a_2-1} \cdots r_2(t_1 - t_2)) &= r_2(r_3r_2) \cdots (r_{a_2-1}r_{a_2-2} \cdots r_3r_2)(t_1 - t_{a_2+1}) \\ &= t_1 - t_{a_2+1} \end{aligned}$$

since the reflection r_j switches t_j and t_{j+1} . Hence we have $p_{s_1}(w) = \pi_{S^1}(t_1 - t_{a_2+1}) = (a_2 + 1 - 1)t = a_2 t = \mathcal{H}_{\mathcal{A}(w)}(1)t$. Now suppose w is 231-type and $i \in [2, a_2]$. The factor in the reduced word decomposition (4.18) corresponding to $[1 = a_1, a_2]$ is equal to $w_{[2, a_2]} \cdot s_1$. By Fact 5.3, for $i > 1$ the presence of the extra s_1 does not affect the Billey computation, so each summand is equal to that for the Peterson type fixed point $w_{[2, a_2]}$ and hence is equal to

$$(i - \mathcal{T}_{\mathcal{A}(w)}(i))t = (i - 1)t,$$

as desired. □

Our next lemma concerns the summands in Billey's formula for $p_{\text{roll}(w)}(w)$ for $w \in \text{Hess}(h)^{S^1}$.

Lemma 5.7. *Let $w \in \text{Hess}(h)^{S^1}$.*

- *Suppose w is 312-type or w is of Peterson type that does not contain 321. Then each summand for Billey's formula for $p_{\text{roll}(w)}(w)$ is equal to*

$$\left(\prod_{i \in \mathcal{A}(w)} (i - \mathcal{T}_{\mathcal{A}(w)}(i) + 1) \right) \cdot t^{|\mathcal{A}(w)|}.$$

- *Suppose w is 231-type. Then each summand for Billey's formula for $p_{\text{roll}(w)}(w)$ is equal to*

$$\mathcal{H}_{\mathcal{A}(w)}(1) \cdot \left(\prod_{i=2}^{\mathcal{H}_{\mathcal{A}(w)}(1)} (i - 1) \right) \cdot \left(\prod_{i \in \mathcal{A}(w) \setminus [\mathcal{T}_{\mathcal{A}(w)}(1), \mathcal{H}_{\mathcal{A}(w)}(1)]} (i - \mathcal{T}_{\mathcal{A}(w)}(i) + 1) \right) t^{|\mathcal{A}(w)|}.$$

- Suppose w is of Peterson type that contains 321. Then each summand for Billey's formula for $p_{\text{roll}(w)}(w)$ is equal to

$$\left(\prod_{i \in \mathcal{A}(w)} (i - \mathcal{T}_{\mathcal{A}(w)}(i) + 1) \right) \cdot t^{|\mathcal{A}(w)|+1}.$$

Proof. Before considering the separate cases we make a general observation. By Lemma 5.6 and the discussion before Lemma 5.6 we know that the summands in Billey's formula for $p_{s_i}(w)$ for $i \in \mathcal{A}(w)$ are exactly the terms $\pi_{S^1}(r(\ell, \mathbf{b}))$ for ℓ such that $b_\ell = i$. Suppose in addition that $w \in \text{Hess}(h)^{S^1}$ is such that $\text{roll}(w)$ contains at most one simple transposition s_i for each $i \in \{1, 2, \dots, n-1\}$, i.e., $\text{roll}(w) = s_{i_1} s_{i_2} \cdots s_{i_{\ell(\text{roll}(w))}}$ is a reduced word for $\text{roll}(w)$ where all i_k are distinct for $1 \leq k \leq \ell(\text{roll}(w))$. This implies that any subword of a reduced word decomposition \mathbf{b} of w which is a reduced word of $\text{roll}(w)$ also must contain precisely one s_{i_k} for each $1 \leq k \leq \ell(\text{roll}(w))$. From Billey's formula (5.3) for $p_{\text{roll}(w)}(w)$ we know that a summand is of the form

$$(5.5) \quad \pi_{S^1}(r(j_1, \mathbf{b})) \pi_{S^1}(r(j_2, \mathbf{b})) \cdots \pi_{S^1}(r(j_{\ell(v)}, \mathbf{b}))$$

where $s_{b_{j_1}} s_{b_{j_2}} \cdots s_{b_{j_{\ell(\text{roll}(w))}}}$ is a reduced word of $\text{roll}(w)$. Since $\{b_{j_1}, \dots, b_{j_{\ell(\text{roll}(w))}}\} = \{i_1, i_2, \dots, i_{\ell(\text{roll}(w))}\}$ for each such summand the quantity (5.5) is equal to

$$(5.6) \quad \prod_{k=1}^{\ell(\text{roll}(w))} p_{s_{i_k}}(w).$$

We now take cases. Suppose w is not a Peterson-type that contains 321. Then from the explicit descriptions of $\text{roll}(w)$ given in Section 4 it follows that $\text{roll}(w)$ contains in its reduced word a single s_i for each $i \in \mathcal{A}(w)$. Thus we are in the situation described in the above paragraph and the claims follow from the computations given in Lemma 5.6.

Suppose w is Peterson type and contains 321. Let \mathbf{b} be the standard reduced word decomposition (cf. (4.13) and (4.15)) of w . We claim that the only reduced word decompositions of $\text{roll}(w)$ that occur as a subword of \mathbf{b} are those which contain two s_1 's, only one s_2 , and precisely one s_{j_ℓ} for all other j_ℓ . Indeed let $\mathcal{A}(w) = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m]$ be the decomposition of $\mathcal{A}(w)$ into maximal consecutive substrings. Recall $a_2 \geq 2$ and $a_1 = 1$ in this case. The rolldown $\text{roll}(w)$ is

$$(s_{a_m} s_{a_m-1} \cdots s_{a_{m-1}}) \cdots (s_{a_4} s_{a_4-1} \cdots s_{a_3}) \cdot (s_{a_2} s_{a_2-1} \cdots s_1 s_2 s_1)$$

and w is

$$w = w_{[a_1, a_2]} w_{[a_3, a_4]} \cdots w_{[a_{m-1}, a_m]}.$$

Let $\ell > 1$. There is only one reduced word decomposition of the factor $s_{a_{\ell+1}} s_{a_{\ell+1}-1} \cdots s_{a_\ell}$ in $\text{roll}(w)$, so it remains to analyze the subwords of $w_{[a_1, a_2]}$ which are reduced words of $s_{a_2} s_{a_2-1} \cdots s_1 s_2 s_1$. Let \mathbf{b} denote the standard reduced word of $w_{[a_1, a_2]}$. Note that another valid reduced word of $s_{a_2} s_{a_2-1} \cdots s_1 s_2 s_1$ is $s_{a_2} s_{a_2-1} \cdots s_2 s_1 s_2$. Since s_2 does not commute with s_1 , the rightmost s_2 in the word $s_{a_2} s_{a_2-1} \cdots s_2 s_1 s_2$ must appear to the right of the s_{a_2} ; in particular, there are two s_2 's to the right of the s_{a_2} in this word. Since there is only one s_2 appearing to the right of the s_{a_2} in \mathbf{b} we conclude that the reduced word of $s_{a_2} s_{a_2-1} \cdots s_1 s_2 s_1$ containing two copies of s_2 never appears as a subword of \mathbf{b} . Hence the only subwords of \mathbf{b} contributing to summands in Billey's formula for $p_{\text{roll}(w)}(w)$ contain two s_1 's and one s_2 , as claimed. Now since $j_1 = 1$ and $j_2 = 2$ and $p_{s_1}(w) = (1 - \mathcal{T}_{\mathcal{A}(w)}(1) + 1)t = (1 - 1 + 1)t = t$ by Lemma 5.6, the claim follows. \square

We have just seen that all summands in Billey's formula for $p_{\text{roll}(w)}(w)$ are equal for all fixed points $w \in \text{Hess}(h)^{S^1}$. In order to finish the computation we must now compute the number of summands which occur.

Lemma 5.8. *Let $w \in \text{Hess}(h)^{S^1}$.*

- Suppose w is of Peterson type that contains 321. Then the number of summands in Billey's formula for $p_{\text{roll}(w)}(w)$ is $\mathcal{H}_{\mathcal{A}(w)}(1) - 1$.
- Suppose w is of Peterson type that does not contain 321. Then the number of summands in Billey's formula for $p_{\text{roll}(w)}(w)$ is 1.
- Suppose w is 312-type. Then the number of summands in Billey's formula for $p_{\text{roll}(w)}(w)$ is $\mathcal{H}_{\mathcal{A}(w)}(1) - 1$.

- Suppose w is 231-type. Then the number of summands in Billey's formula for $p_{\text{roll}(w)}(w)$ is 1.

Proof. We consider each case in turn. Suppose w is of Peterson type that contains 321. Let $\mathcal{A}(w) = [a_1, a_2] \cup [a_3, a_4] \cup \dots \cup [a_{m-1}, a_m]$ be the decomposition of $\mathcal{A}(w)$ into maximal consecutive substrings. Recall $a_2 \geq 2$ and $a_1 = 1$ in this case. The rolldown $\text{roll}(w)$ is

$$(s_{a_m} s_{a_m-1} \cdots s_{a_{m-1}}) \cdots (s_{a_4} s_{a_4-1} \cdots s_{a_3}) \cdot (s_{a_2} s_{a_2-1} \cdots s_1 s_2 s_1)$$

and w is

$$w = w_{[a_1, a_2]} w_{[a_3, a_4]} \cdots w_{[a_{m-1}, a_m]}.$$

Let $\ell > 1$. As observed in the proof of Lemma 5.7 there is only one reduced word decomposition of the factor $s_{a_{\ell+1}} s_{a_{\ell+1}-1} \cdots s_{a_\ell}$ in $\text{roll}(w)$. Moreover by examination it is evident that it appears only once in the standard reduced word decomposition of the corresponding $w_{[a_\ell, a_{\ell+1}]}$ factor in w . Hence in order to count the number of ways $\text{roll}(w)$ appears in w it suffices to count the number of subwords of the standard reduced word decomposition \mathbf{b} of $w_{[a_1, a_2]}$ which are reduced subwords of $s_{a_2} s_{a_2-1} \cdots s_1 s_2 s_1$. We already saw in the proof of Lemma 5.7 that the reduced word $s_{a_2} s_{a_2-1} \cdots s_2 s_1 s_2$ never appears in \mathbf{b} . On the other hand since s_1 commutes with any s_k with $k \geq 3$, another reduced word decomposition of $s_{a_2} s_{a_2-1} \cdots s_1 s_2 s_1$ is $s_1 s_{a_2} s_{a_2-1} \cdots s_2 s_1$. From examination of \mathbf{b} it can be seen that the word $s_1 s_{a_2} s_{a_2-1} \cdots s_2 s_1$ appears as a subword in the standard reduced word of $w_{[a_1, a_2]}$ precisely $a_2 - 1 = \mathcal{H}_{\mathcal{A}(w)}(1) - 1$ times and that these are the only subwords of \mathbf{b} which equal $s_{a_2} s_{a_2-1} \cdots s_1 s_2 s_1$. The claim follows.

Suppose w is of Peterson type that does not contain 321. Then the rolldown $\text{roll}(w)$ is the Peterson case rolldown so the claim follows from explicit examination of the standard reduced word of w (alternatively from [11, Fact 4.5]).

Suppose w is 312-type. Then the rolldown $\text{roll}(w)$ is of the form

$$\text{roll}(w) = (s_{a_m} s_{a_m-1} \cdots s_{a_{m-1}}) \cdots (s_{a_4} s_{a_4-1} \cdots s_{a_3}) \cdot (s_{a_2} s_{a_2-1} \cdots s_1 s_2)$$

from Lemma 4.20. By an argument similar to the case of Peterson type that contains 321 it suffices to analyze only the factors in both $\text{roll}(w)$ and w corresponding to the initial maximal consecutive substring $[a_1, a_2]$. As above we have

$$s_{a_2} s_{a_2-1} \cdots s_1 s_2 = s_1 s_{a_2} s_{a_2-1} \cdots s_2$$

and again it follows from examination of the standard reduced word of $u_{[a_1, a_2]}$ that $s_1 s_{a_2} s_{a_2-1} \cdots s_2$ appears precisely $a_2 - 1 = \mathcal{H}_{\mathcal{A}(w)}(1) - 1$ times.

Finally suppose w is 231-type. Then the rolldown $\text{roll}(w)$ coincides with the Peterson case rolldown of $w_{\mathcal{A}(w)}$ and the claim follows from examination of the reduced word decomposition (4.18). \square

The following is immediate from Lemmas 5.7 and 5.8.

Proposition 5.9. Let $w \in \text{Hess}(h)^{S^1}$.

- Suppose w is of Peterson type that contains 321. Then

$$p_{\text{roll}(w)}(w) = (\mathcal{H}_{\mathcal{A}(w)}(1) - 1) \left(\prod_{i \in \mathcal{A}(w)} (i - \mathcal{T}_{\mathcal{A}(w)}(i) + 1) \right) \cdot t^{|\mathcal{A}(w)|+1}.$$

- Suppose w is of Peterson type that does not contain 321. Then

$$p_{\text{roll}(w)}(w) = \left(\prod_{i \in \mathcal{A}(w)} (i - \mathcal{T}_{\mathcal{A}(w)}(i) + 1) \right) t^{|\mathcal{A}(w)|}.$$

- Suppose w is of type 312. Then

$$p_{\text{roll}(w)}(w) = (\mathcal{H}_{\mathcal{A}(w)}(1) - 1) \left(\prod_{i \in \mathcal{A}(w)} (i - \mathcal{T}_{\mathcal{A}(w)}(i) + 1) \right) \cdot t^{|\mathcal{A}(w)|}.$$

- Suppose w is of type 231. Then

$$p_{\text{roll}(w)}(w) = \mathcal{H}_{\mathcal{A}(w)}(1) \cdot \left(\prod_{i=2}^{\mathcal{H}_{\mathcal{A}(w)}(1)} (i - 1) \right) \cdot \left(\prod_{i \in \mathcal{A}(w) \setminus [\mathcal{T}_{\mathcal{A}(w)}(1), \mathcal{H}_{\mathcal{A}(w)}(1)]} (i - \mathcal{T}_{\mathcal{A}(w)}(i) + 1) \right) t^{|\mathcal{A}(w)|}.$$

The proofs of the main results are now immediate.

Proof of Proposition 4.2. Let $w \in \text{Hess}(h)^{S^1}$. From the explicit formulas given in Proposition 5.9 it follows that $p_{\text{roll}(w)}(w) \neq 0$ for all possible types of fixed points w . \square

Proof of Theorem 4.1. Since both (4.3) and (4.4) are satisfied for all $w, w' \in \text{Hess}(h)^{S^1}$ by Propositions 4.3 and 4.2 respectively, the result follows. \square

6. OPEN QUESTIONS

This manuscript raises more questions than it answers. We close by mentioning some of them.

Question 6.1. For $n \geq 4$, Theorem 4.1 shows that for the case when $N : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is the principal nilpotent operator and h is the 334-type Hessenberg function, the dimension pair algorithm produces a set of Hessenberg Schubert classes $\{p_{\text{roll}(w)}\}_{w \in \text{Hess}(N, h)^{S^1}}$ which are poset-upper-triangular and hence form a $H_{S^1}^*(\text{pt})$ -module basis for $H_{S^1}^*(\text{Hess}(N, h))$.

- (1) What are other examples of N and h such that the conclusion of Lemma 3.4 holds (cf. Remark 3.8)?
- (2) What are other examples of N and h for which the dimension pair algorithm produces a successful outcome of Betti poset pinball which is also poset-upper-triangular? Are there necessary and sufficient conditions on N and h that guarantee poset-upper-triangularity?
- (3) What are other examples of N and h for which the dimension pair algorithm produces a successful outcome of Betti poset pinball which corresponds to a linearly independent set of classes and hence a module basis? Are there necessary and sufficient conditions on N and h that guarantee this?

Question 6.2. In [11] the explicit module basis consisting of Peterson Schubert classes is used to derive a **manifestly positive Monk formula** in the S^1 -equivariant cohomology of Peterson varieties. Preliminary investigation suggests that an analogous Monk formula for the 334-type Hessenberg varieties, using the module basis of Hessenberg Schubert classes derived in this manuscript, would be computationally much more complex. Thus, we may ask the following.

- (1) Does there exist a combinatorially elegant or computationally effective Monk formula for the 334-type Hessenberg varieties?
- (2) Can such a Monk formula be further generalized to a larger family of regular nilpotent Hessenberg varieties? For instance, can our techniques be generalized to give new insights to the equivariant Schubert calculus of the full flag variety $\text{Flags}(\mathbb{C}^n)$ (which is an example of a regular nilpotent Hessenberg variety)?
- (3) In [1] the Monk formula for Peterson varieties is used to derive a **Giambelli formula**. Does there also exist a combinatorially elegant and/or computationally effective Giambelli formula for other cases of regular nilpotent Hessenberg varieties?

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DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, CENTRE FOR MATHEMATICAL SCIENCES, WILBERFORCE ROAD, CAMBRIDGE CB3 0WA, UNITED KINGDOM

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCMASTER UNIVERSITY, 1280 MAIN STREET WEST, HAMILTON, ONTARIO L8S4K1, CANADA